BOUNDARY VALUE PROBLEM
FOR THE RADIATION TRANSFER EQUATION WITH REFLECTION AND REFRACTION CONDITIONS

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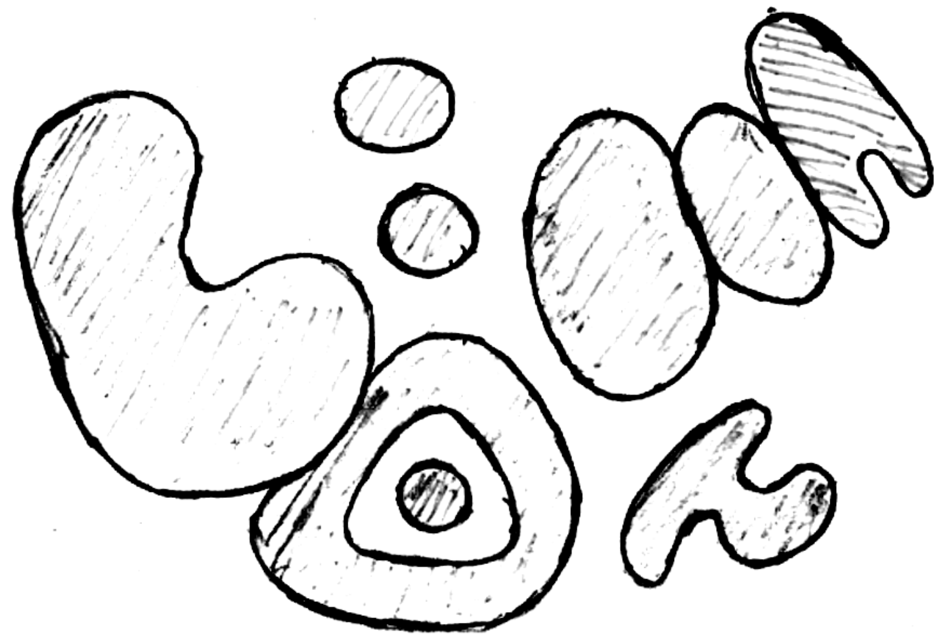
Introduction

We consider the monochormatic radiation transfer in the system $G = \bigcup_{j=1}^{m} G_j$ of semitransparent bodies G_j separated by the vacuum. Each body G_j is a bounded domain in $\mathbb{R}^3$ with boundary ∂G_j of class C^1. We assume that the domains G_i and G_j are pairwise disjoint, whereas the boundaries of G_i and G_j can intersect for some $i \neq j$.

Assume that each G_j is occupied by a semitransparent medium with constant absorption $\varkappa_j > 0$ and scattering $s_j \geqslant 0$ coefficients and the refraction exponent $k_j > 1$. We set $\varkappa(x) = \varkappa_j$, $s(x) = s_j$, and $k(x) = k_j$ for $x \in G_j$, $1 \leqslant j \leqslant m$.

Let $\Omega = \{\omega \in \mathbb{R}^3 \mid |\omega| = 1\}$ be the unit sphere in $\mathbb{R}^3$ (the sphere of directions).

The sought function $I(\omega, x)$ is defined on the set $D = \Omega \times G$ and is interpreted as the radiation intensity at a point $x \in G$ when the radiation propagates along the direction $\omega \in \Omega$.



System of semitransparent bodies $G = \bigcup_{j=1}^{m} G_j$

To describe the radiation propagation in G, we use **the radiation transfer equation**

$$\omega \cdot \nabla I + (s + \varkappa) I = s\mathscr{S}(I) + \varkappa k^2 F, \quad (\omega, x) \in D,$$

where

$$\omega \cdot \nabla I = \sum_{i=1}^{3} \omega_i \frac{\partial}{\partial x_i} I$$

denotes the derivative of a function I along the direction ω and $\mathscr{S}$ denotes the scattering operator

$$\mathscr{S}(I)(\omega, x) = \frac{1}{4\pi} \int\limits_{\Omega} \theta_j(\omega' \cdot \omega) I(\omega')\, d\omega', \ (\omega, x) \in D_j = \Omega \times G_j, \ 1 \leqslant j \leqslant m$$

with the scattering indicatrix possessing the following properties:

$$\theta_j \in L^1(-1,1), \quad \theta_j \geqslant 0, \quad \frac{1}{2} \int\limits_{-1}^{1} \theta_j(\mu)\, d\mu = 1, \quad 1 \leqslant j \leqslant m.$$

Function $F(\omega, x)$ characterizes the density of radiation of volume sources.

The equation of radiation transfer governs different physical phenomena. The mathematical properties of this equation were studied by many authors. We mention only the classical work and monographs

V. S. Vladimirov, "Mathematical problems in the theory of single-velocity particle transfer" [in Russian], Trudy MIAN SSSR 61, 3-158 (1961).

T. A. Germogenova, Local Properties of Solutions of the Transport Equation [in Russian], Nauka, Moscow (1986).

V. I. Agoshkov, Boundary Value Problems for Transport Equations: Functional Spaces, Variational Statements, Regularity of Solutions, Birkhauser, Basel etc. (1998).

The most studied problems in this relation are boundary value problems for the equation of radiation transfer with the continuity condition for the radiation intensity imposed on the interface between media with different optic properties. In this case, the radiation passes through the interface without changing direction and intensity. In the case of nonconvex domains, this condition leads to the "shooting condition". In some applications (for example, the theory of neutron transfer), such conditions are justified from the physical point of view.

However, in many applications (for example, optics, tomography, thermal physics), the reflection and refraction of radiation at the interface between media should be taken into account. It often happens that the reflection and refraction cause important physical effects determining the content of the problem under consideration.

The boundary value problems for the equation of radiation transfer with the reflection and refraction conditions at the interface are still studied unsatisfactory, in spite of the importance of such problems in applications. Some problems with conditions taking into account reflection on the interface between media were treated in

T. A. Germogenova, Local Properties of Solutions of the Transport Equation [in Russian], Nauka, Moscow (1986).

T. A. Germogenova, Gerneralized solutions of boundary value problems for the transfer equation, U.S.S.R. Comput. Math. Math. Phys. 9, No. 3, 139-166 (1971).

From the mathematical point of view, problems for the equation of radiation transfer with the reflection and refraction conditions in accordance with the Fresnel laws were first studied in the papers

I. V. Prokhorov, "Boundary value problem of radiation transfer in an inhomogeneous medium with reflection conditions on the boundary", Differ. Equ. 36, No. 6, 943-948 (2000).

I. V. Prokhorov, "On the solvability of a boundary value problem in radiation transfer theory with generalized conjugation conditions at the interface between media", Izv. Math. 67, No. 6, 1243-1266 (2003).

We will discuss some new results published in 2013 year:

Amosov A.A. Boundary value problem for the radiation transfer equation with reflection and refraction conditions // Journal of Mathematical Sciences. 2013. Vol. 191. No 2. p. 101–149

Amosov A.A. Boundary value problem for the radiation transfer equation with diffuse reflection and refraction conditions // Journal of Mathematical Sciences. 2013. Vol. 193, No 2, p. 151–176.

Amosov A.A. Boundary value problem for the radiation transfer equation with reflection and refraction conditions. Continuous dependence of solutions from data and the limiting transition to the prooblem with "shooting conditions" // Journal of Mathematical Sciences. 2013. to appear.

1 Some notions and functional spaces

Remind that $D = \Omega \times G$. We denote by $\mathscr{W}^p(D)$ the Banach space of functions possessing the weak directional derivative $\omega \cdot \nabla f \in L^p(D)$ with the norm

$$\|f\|_{\mathscr{W}^p(D_j)} = \begin{cases} \left(\|f\|^p_{L^p(D_j)} + \|\omega \cdot \nabla f\|^p_{L^p(D_j)}\right)^{1/p}, & 1 \leqslant p < \infty, \\ \max\{\|f\|_{L^\infty(D_j)}, \|\omega \cdot \nabla f\|_{L^\infty(D_j)}\}, & p = \infty \end{cases}$$

Let

$$\Gamma_j = \Omega \times \partial G_j, \quad \Gamma = \Omega \times \partial G = \bigcup_{j=1}^{m} \Gamma_j,$$

$$\Gamma_j^+ = \{(\omega, x) \in \Gamma_j \mid \omega \cdot n_j(x) > 0\}, \quad \Gamma^+ = \bigcup_{j=1}^{m} \Gamma_j^+,$$

$$\Gamma_j^- = \{(\omega, x) \in \Gamma_j \mid \omega \cdot n_j(x) < 0\}, \quad \Gamma^- = \bigcup_{j=1}^{m} \Gamma_j^-.$$

Let $d\omega$ and $d\sigma(x)$ be the measures on Ω and ∂G induced by the Lebesgue measure in $\mathbb{R}^3$.

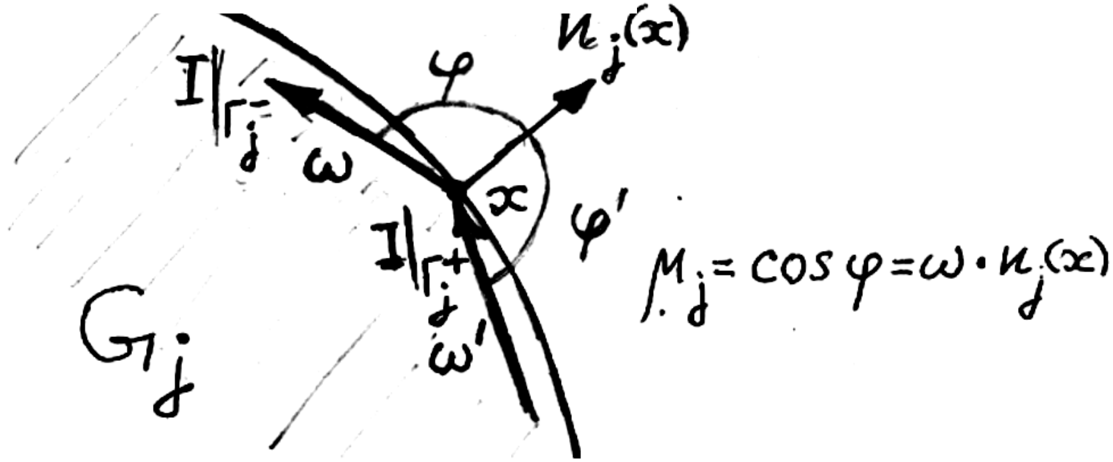
Let $E^\pm$ be a measurable subset of the set $\Gamma^\pm$. Introduce the Banach spaces $\widehat{L}^p(E^\pm)$ of functions g that are defined on $E^\pm$, measurable with respect to the measure $d\omega d\sigma(x)$ and have finite norm

$$\|g\|_{\widehat{L}^p(E^\pm)} = \begin{cases} \left(\int\limits_{E^\pm} |g(\omega, x)|^p \, |\omega \cdot n_j(x)| d\omega d\sigma(x)\right)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}\limits_{(\omega,x)\in E^\pm} |g(\omega, x)|, & p = \infty. \end{cases}$$

2 Boundary conditions

We state conditions for the radiation intensity on different parts of the boundary. For the radiation propagating in G the values of its intensity on the sets $\Gamma^{\pm}$ and $\Gamma_j^{\pm}$ are denoted by $I|_{\Gamma^{\pm}}$ and $I|_{\Gamma_j^{\pm}}$ respectively, whereas for the radiation propagating in the vacuum the values of its intensity on the set Γ are denoted by J.

2.1 Condition of complete inner reflection.



Suppose that the radiation with intensity $I(\omega', x)$ propagating inside the domain G_j falls to a point x of the surface ∂G_j in the direction ω' under the incidence angle φ' with $\cos\varphi' = \mu_j' = \omega' \cdot n_j(x) > 0$.

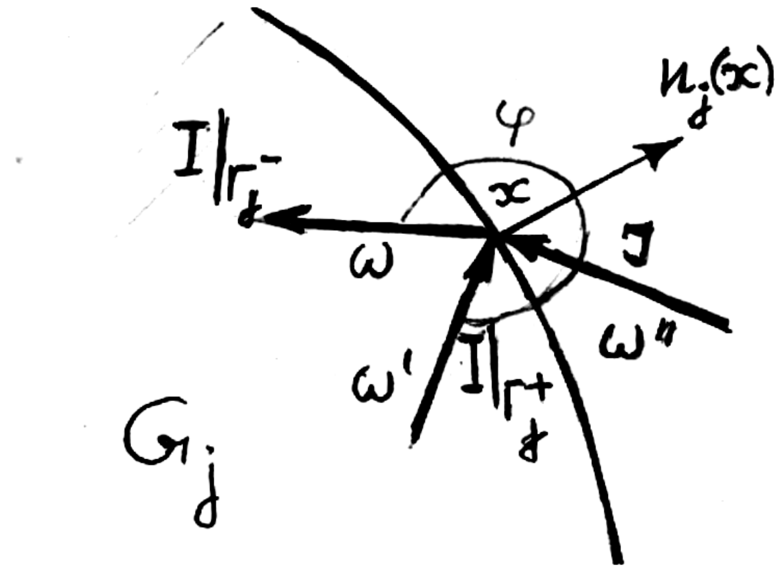
If $0 < \mu_j' \leqslant \sqrt{1 - 1/k_j^2}$, then the **effect of complete inner reflection holds**: the radiation is completely reflected and propagates in the direction $\omega = \omega - 2\mu_j' n_j(x)$ under the reflection angle φ with $\cos\varphi = \mu_j = \omega \cdot n_j(x) = -\mu_j' < 0$. Thus,

$$I|_{\Gamma_j^-} = \mathscr{R}^-(I|_{\Gamma_j^+}), \quad \text{if} \quad -\sqrt{1 - 1/k_j^2} \leqslant \mu_j < 0.$$

where

$$\mathscr{R}^-(I|_{\Gamma_j^+})(\omega, x) = I|_{\Gamma_j^+}(\omega - 2\mu_j n_j(x), x).$$

2.2 Condition of reflection and refraction for I.



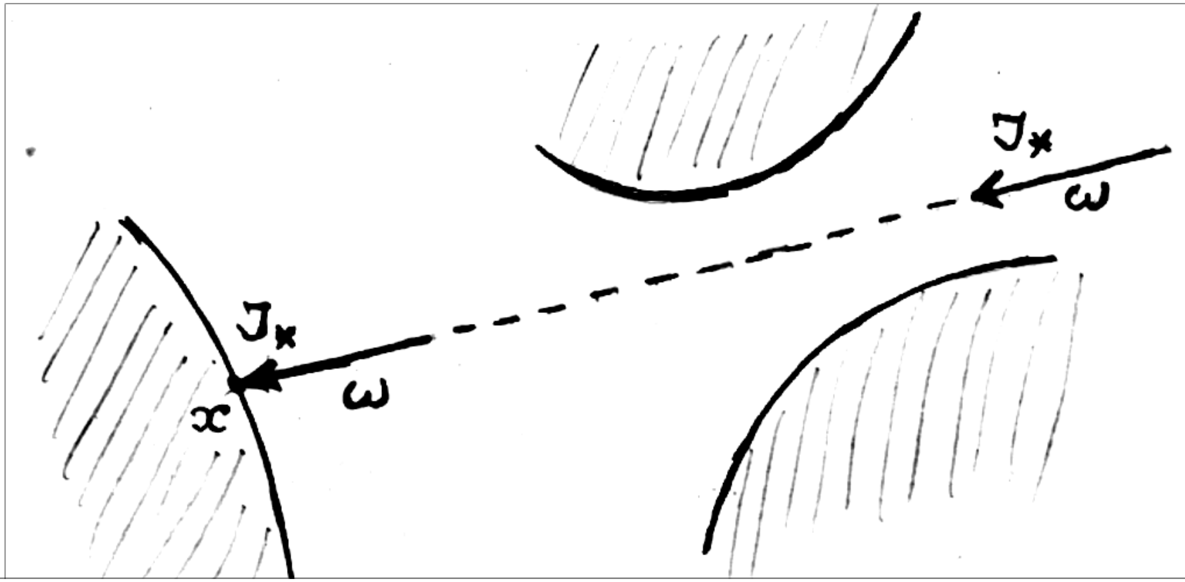
If $-1 \leqslant \mu_j < -\sqrt{1-1/k_j^2}$, then incoming radiation consists of partially reflected and partially refracted radiations:

$$I|_{\Gamma_j^-} = \mathscr{R}^-(I|_{\Gamma_j^+}) + \mathscr{P}^-(J), \quad \text{if} \quad -1 \leqslant \mu_j < -\sqrt{1-1/k_j^2},$$

where

$$\mathscr{R}^-(I|_{\Gamma_j^+})(\omega, x) = r_j^-(\mu_j) I|_{\Gamma_j^+}(\omega'), \quad \omega' = \omega - 2\mu_j n_j(x),$$
$$\mathscr{P}^-(J)(\omega, x) = (1 - r_j^-(\mu_j)) k_j^2 J(\omega'', x),$$
$$\omega'' = -\nu_j^+(\mu_j) n_j(x) + k_j(\omega - \mu_j n_j(x)), \quad \nu_j^+(\mu_j) = \sqrt{1 - k_j^2(1-\mu^2)},$$
$$r_j^-(\mu_j) = \frac{1}{2}\left[\left(\frac{\nu_j^+(\mu_j) + k_j\mu_j}{\nu_j^+(\mu_j) - k_j\mu_j}\right)^2 + \left(\frac{k_j\nu_j^+(\mu_j) + \mu_j}{k_j\nu_j^+(\mu_j) - \mu_j}\right)^2\right]$$

2.3 Condition for falling from vacuum and coming from outside radiation J



Let

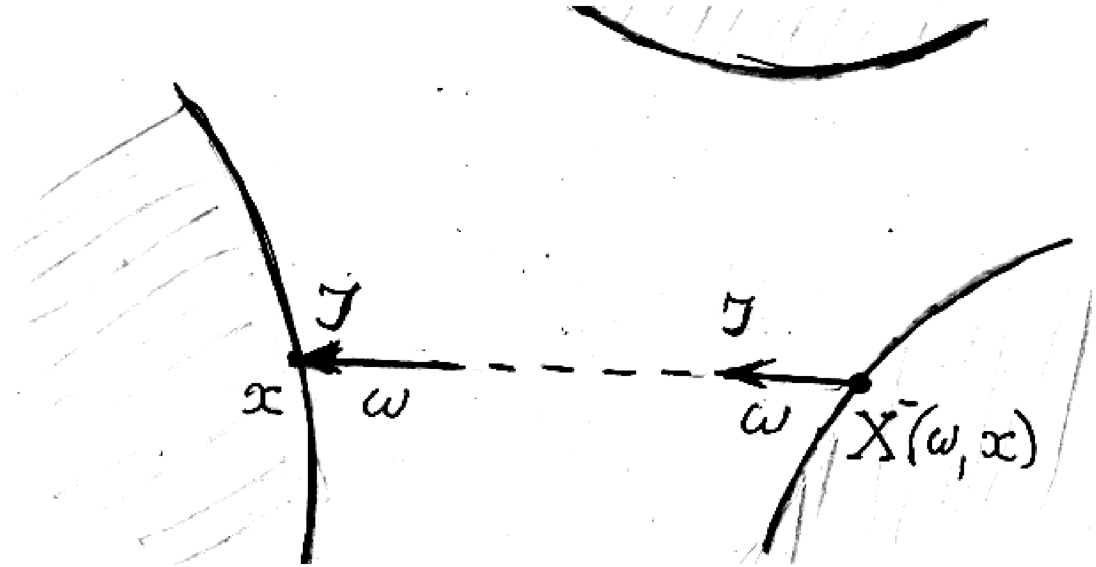
$$S_j^- = \{(\omega, x) \in \Gamma_j^- \mid x \in \partial G_j \setminus \bigcup_{i \neq j} \partial G_i\}, \quad S^- = \bigcup_{j=1}^m S_j^-,$$

$$\overset{*}{S}{}_j^- = \{(\omega, x) \in S_j^- \mid \{x - t\omega,\ t > 0\} \cap \overline{G} = \emptyset\}, \quad \overset{*}{S}{}^- = \bigcup_{j=1}^m \overset{*}{S}{}_j^-.$$

For $(\omega, x) \in \overset{*}{S}{}^-$ the incident radiation J falling from the vacuum to G goes outside and we can assume that it is prescribed:

$$J = J_*, \quad (\omega, x) \in \overset{*}{S}{}^-.$$

2.4 Condition for falling from vacuum and coming from ∂G radiation J



Let

$$\widetilde{S}_j^- = \{(\omega, x) \in S_j^- \mid \{x - t\omega,\ t > 0\} \cap \overline{G} \neq \emptyset\}, \quad \widetilde{S}^- = \bigcup_{j=1}^{m} \widetilde{S}_j^-$$

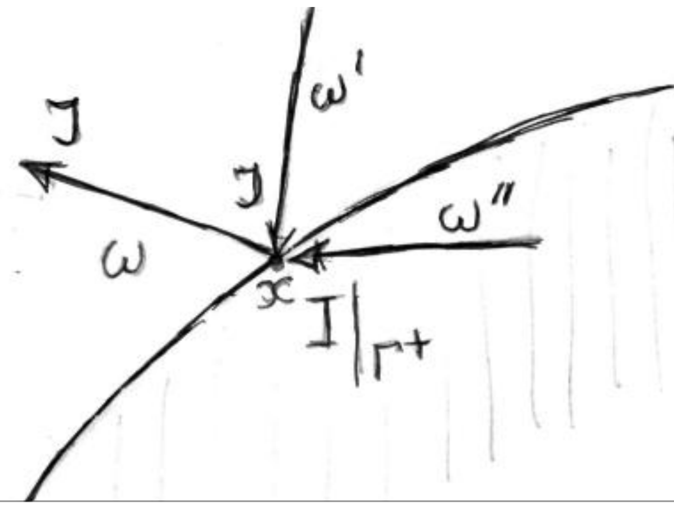
For $(\omega, x) \in \widetilde{S}^-$ the incident radiation J falling from the vacuum to G comes from the point $X^-(\omega, x) \in \partial G$:

$$J = T(J), \quad (\omega, x) \in \widetilde{S}^-,$$

where

$$T(J)(\omega, x) = J(\omega, X^-(\omega, x)),$$
$$X^-(\omega, x) = x - \tau^-(\omega, x)\omega, \quad \tau^-(\omega, x) = \inf\{t > 0 \mid x - t\omega \in \overline{G}\}.$$

2.5 Condition of reflection and refraction for J.



Let $(\omega, x) \in \Gamma_j^+$ and $x \in \partial G_j \setminus \Sigma_j$. Then

$$J = \mathscr{R}^+(J) + \mathscr{P}^+(I|_{\Gamma_j^+}),$$

where

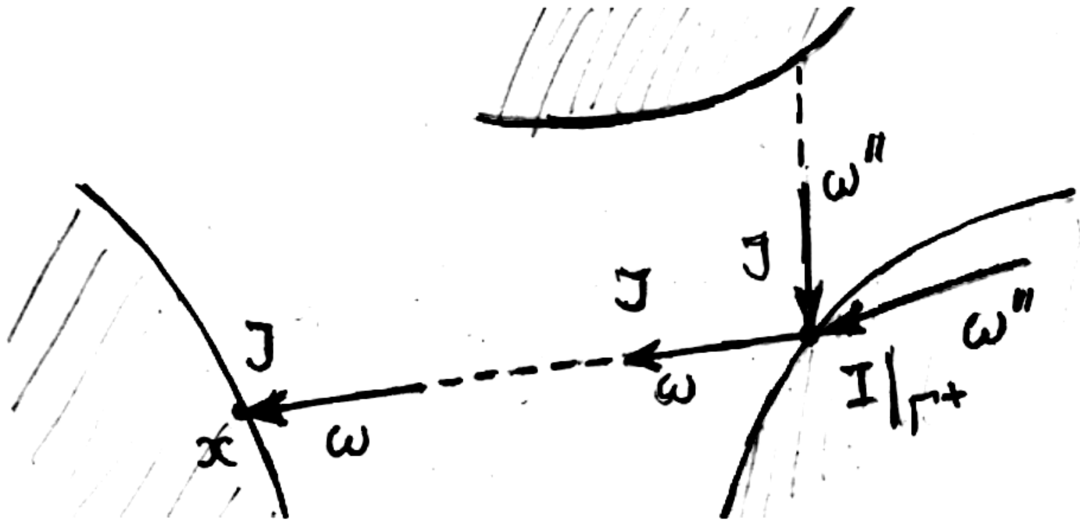
$$\mathscr{R}^+(J)(\omega, x) = r_j^+(\mu_j) J(\omega', x), \quad \omega' = \omega - 2\mu_j n_j(x),$$

$$\mathscr{P}^+(I_{\Gamma_j^+})(\omega, x) = (1 - r_j^+(\mu_j)) \frac{1}{k_j^2} I_{\Gamma_j^+}(\omega'', x),$$

$$\omega'' = \nu_j^-(\mu_j) n_j(x) + \frac{1}{k_j}(\omega - \mu_j n_j(x)), \quad \nu_j^-(\mu_j) = \sqrt{1 - \frac{1}{k_j^2}(1 - \mu^2)},$$

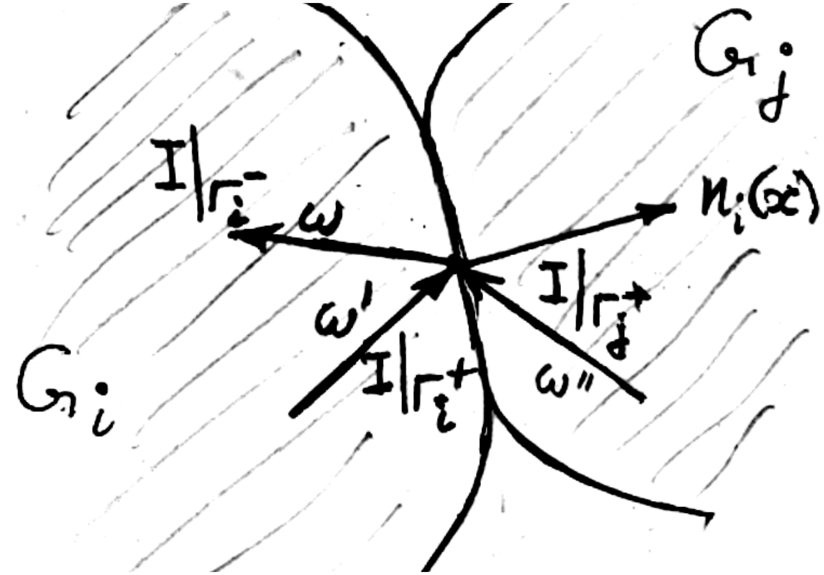
$$r_j^+(\mu_j) = \frac{1}{2}\left[\left(\frac{\mu_j - k_j \nu^-(\mu_j)}{\mu_j + k_j \nu^-(\mu_j)}\right)^2 + \left(\frac{k_j \mu_j - \nu_j^-(\mu_j)}{k_j \mu_j + \nu_j^-(\mu_j)}\right)^2\right].$$

2.6 Condition for falling from vacuum, coming from ∂G reflected and refracted radiation J



$$J = T\mathscr{R}^{+}(J) + T\mathscr{P}^{+}(I|_{\Gamma_j^+}), \quad (\omega, x) \in \widetilde{S}^{-}.$$

2.7 Condition of reflection and refraction on the interface of two bodies



$$I|_{\Gamma_i^-} = \mathscr{R}_{ij}^-(I|_{\Gamma_i^+}) + \mathscr{P}_{ij}^-(I|_{\Gamma_j^+}),$$

where

$$\mathscr{R}_{ij}^-(I_{\Gamma_i^+})(\omega, x) = r_{ij}^-(\mu_i) I_{\Gamma_i^+}(\omega', x),$$

$$\mathscr{P}_{ij}^-(I_{\Gamma_j^+})(\omega, x) = (1 - r_{ij}^-(\mu_i)) \frac{k_i^2}{k_j^2} I_{\Gamma_j^+}(\omega'', x), x),$$

$$\omega' = \omega - 2\mu_i n_i(x), \quad \omega'' = -\nu_{ij}^+(\mu_i) n_i(x) + \frac{k_i}{k_j}(\omega - \mu_i n_i(x)), \quad \mu_i = \omega \cdot n_i(x),$$

$$\nu_{ij}^+(\mu_i) = \sqrt{1 - \frac{k_i^2}{k_j^2}(1 - \mu_i^2)},$$

$$r_{ij}^-(\mu_i) = \frac{r_i^-(\mu_i) + r_j^+(\nu_i^+(\mu_i)) - 2 r_i^-(\mu_i) r_j^+(\nu_i^+(\mu_i))}{1 - r_i^-(\mu_i) r_j^+(\nu_i^+(\mu_i))}.$$

3 Boundary value problem for the radiation transfer equation with reflection and refraction conditions

Consider the boundary value problem

$$\omega \cdot \nabla I + (\varkappa + s) I = s\mathscr{S}(I) + \varkappa k^2 F, \quad (\omega, x) \in D, \tag{1}$$

$$I|_{\Gamma^-} = \mathscr{R}^-(I|_{\Gamma^+}) + \mathscr{P}^-(J), \quad (\omega, x) \in S^-, \tag{2}$$

$$I|_{\Gamma_i^-} = \mathscr{R}_{ij}^-(I|_{\Gamma_i^+}) + \mathscr{P}_{ij}^-(I|_{\Gamma_j^+}), \quad (\omega, x) \in \Gamma_i^- \cap \Gamma_j^+, \quad i \neq j, \tag{3}$$

$$J = T\mathscr{R}^+(J) + T\mathscr{P}^+(I|_{\Gamma^+}), \quad (\omega, x) \in \widetilde{S}^-, \tag{4}$$

$$J = J_*, \quad (\omega, x) \in \overset{*}{S}{}^-. \tag{5}$$

Assume that $F \in L^1(D)$, $J_* \in \widehat{L}^1(\overset{*}{S}{}^-)$.

By a solution to the problem (1)–(5) we mean a function $I \in \mathscr{W}^1(D)$ that satisfies equation (1) almost everywhere on D and the conditions (2), (3) almost everywhere on S^-, $\bigcup\limits_{i\neq j}(\Gamma_i^- \cap \Gamma_j^+)$. At the same time function $J \in \widehat{L}^1_{1-r^+}(S^-)$ satisfies conditions (4) и (5) almost everywhere on $\widetilde{S}^-$ and $\overset{*}{S}{}^-$.

Theorem 3.1. *A solution to the problem (1) – (5) $I \in \mathscr{W}^1(D)$ exists and is unique. If additionally $F \in L^p(D)$, $J_* \in \widehat{L}^p(\overset{*}{S}{}^-)$ with some $p \in (1, \infty]$ then $I \in \mathscr{W}^p(D)$.*

For $1 \leqslant p < \infty$ the solution to the problem (1)–(5) satisfies the estimates

$$\|\varkappa^{1/p} k^{2/p-2} I\|_{L^p(D)} \leqslant \left(\|\varkappa^{1/p} k^{2/p} F\|^p_{L^p(D)} + \|J_*\|^p_{\widehat{L}^p(\overset{*}{S}{}^-)} \right)^{1/p},$$

$$\|\varkappa^{1/p-1} k^{2/p-2}\, \omega \cdot \nabla I\|_{L^p(D)} \leqslant \frac{2}{1 - \varpi_{\max}} \left(\|\varkappa^{1/p} k^{2/p} F\|^p_{L^p(D)} + \|J_*\|^p_{\widehat{L}^p(\overset{*}{S}{}^-)} \right)$$

For $p = \infty$ it satisfies the estimates

$$\|k^{-2} I\|_{L^\infty(D)} \leqslant \max\{\|F\|_{L^\infty(D)}, \|J_*\|_{L^\infty(\overset{*}{S}{}^-)}\},$$

$$\|\varkappa^{-1} k^{-2}\, \omega \cdot \nabla I\|_{L^\infty(D)} \leqslant \frac{2}{1 - \varpi_{\max}} \max\{\|F\|_{L^\infty(D)}, \|J_*\|_{L^\infty(\overset{*}{S}{}^-)}\}.$$

Here $\varpi_{\max} = \max\limits_{1 \leqslant j \leqslant m} \dfrac{s_j}{\varkappa_j + s_j} < 1$.

4 Boundary value problem for the radiation transfer equation with diffuse reflection and refraction conditions

We considered also the problem

$$
\begin{gathered}
\omega \cdot \nabla I + (\varkappa + s) I = s\mathscr{S}(I) + \varkappa k^2 F, \quad (\omega, x) \in D, \\
I|_{\Gamma^-} = \mathscr{R}_d^-(I|_{\Gamma^+}) + \mathscr{P}_d^-(J), \quad (\omega, x) \in S^-, \\
I|_{\Gamma_i^-} = \mathscr{R}_{d,ij}^-(I|_{\Gamma_i^+}) + \mathscr{P}_{d,ij}^-(I|_{\Gamma_j^+}), \quad (\omega, x) \in \Gamma_i^- \cap \Gamma_j^+, \quad i \neq j, \\
J = T\mathscr{R}_d^+(J) + T\mathscr{P}_d^+(I|_{\Gamma^+}), \quad (\omega, x) \in \widetilde{S}^-, \\
J = J_*, \quad (\omega, x) \in \overset{*}{S}{}^-,
\end{gathered}
$$

describing the radiation transfer in the system of semitransparent bodies with the diffuse reflection and refraction conditions. Here $\mathscr{R}_d^-$, $\mathscr{R}_d^+$, $\mathscr{R}_{d,ij}^+$, $\mathscr{P}_d^-$, $\mathscr{P}_d^+$, $\mathscr{P}_{d,ij}^+$ are operators of diffuse reflection and refraction

We have proved the theorem of the existence and uniqueness of solution to this problem, which is similar in formulation to Theorem 3.1.

5 Continuous dependance of solutions on data

Consider the sequence of boundary value problems

$$\omega\cdot\nabla I^{(n)}+(\varkappa^{(n)}+s^{(n)})I^{(n)}=s^{(n)}\mathscr{S}^{(n)}(I^{(n)})+\varkappa^{(n)}(k^{(n)})^2F^{(n)},\ (\omega,x)\in D, \tag{6}$$

$$I^{(n)}|_{\Gamma^-}=\mathscr{R}^{-,(n)}(I^{(n)}|_{\Gamma^+})+\mathscr{P}^{-,(n)}(J^{(n)}),\quad (\omega,x)\in S^-, \tag{7}$$

$$I^{(n)}|_{\Gamma_i^-}=\mathscr{R}_{ij}^{-,(n)}(I^{(n)}|_{\Gamma_i^+})+\mathscr{P}_{ij}^{-,(n)}(I^{(n)}|_{\Gamma_j^+}),\quad (\omega,x)\in\Gamma_i^-\cap\Gamma_j^+,\ i\neq j, \tag{8}$$

$$J^{(n)}=T\mathscr{R}^{+,(n)}(J^{(n)})+T\mathscr{P}^{+,(n)}(I^{(n)}|_{\Gamma^+}),\quad (\omega,x)\in\widetilde{S}^-, \tag{9}$$

$$J^{(n)}=J_*^{(n)},\quad (\omega,x)\in\overset{*}{S}{}^- \tag{10}$$

with data $\{\varkappa_j^{(n)}\}_{n=1}^\infty$, $\{s_j^{(n)}\}_{n=1}^\infty$, $\{k_j^{(n)}\}_{n=1}^\infty$, $\{\theta_j^{(n)}\}_{n=1}^\infty$, $1\leqslant j\leqslant m$ and $\{F^{(n)}\}_{n=1}^\infty\subset L^1(D)$, $\{J_*^{(n)}\}_{n=1}^\infty\subset\widehat{L}^1(\overset{*}{S}{}^-)$. Assume that

$$\varkappa_j=\lim_{n\to\infty}\varkappa_j^{(n)},\ s_j=\lim_{n\to\infty}s_j^{(n)},\ k_j=\lim_{n\to\infty}k_j^{(n)},\quad \lim_{n\to\infty}\|\theta_j^{(n)}-\theta_j\|_{L^1(-1,1)}=0,\quad 1\leqslant j\leqslant m;$$

$$\lim_{n\to\infty}\|F^{(n)}-F\|_{L^1(D)}=0,\quad \lim_{n\to\infty}\|J_*^{(n)}-J_*\|_{\widehat{L}^1(\overset{*}{S}{}^-)}=0.$$

Theorem 5.1. *Let $\{I^{(n)}\}_{n=1}^\infty$ be a sequence of solutions of problems (6) – (10) and a I be a solution of problem (1) – (5). Then $I^{(n)}\to I$ in $\mathscr{W}^1(D)$ as $n\to\infty$.*

Corollary 5.1. *If additionally $F\in L^p(D)$, $J_*\in\widehat{L}^p(\overset{*}{S}{}^-)$, $\{F^{(n)}\}_{n=1}^\infty\subset L^p(D)$, $\{J_*^{(n)}\}_{n=1}^\infty\subset\widehat{L}^p(\overset{*}{S}{}^-)$ with some $p\in(1,\infty]$ and $\sup_{n\geqslant1}\|F^{(n)}\|_{L^p(D)}<\infty$, $\sup_{n\geqslant1}\|J_*^{(n)}\|_{\widehat{L}^p(\overset{*}{S}{}^-)}<\infty$.*

Then $I^{(n)}\to I$ in $\mathscr{W}^q(D)$ as $n\to\infty$ for all $q\in[1,p)$.

6 Limiting transition to the problem with "shooting conditions"

Theorem 6.1. *Let* $\{I^{(n)}\}_{n=1}^{\infty}$ *be a set of solutions of problems (6) – (10). If refraction exponents* $k_j^{(n)} \to 1$, $1 \leqslant j \leqslant m$ *as* $n \to \infty$, *then* $I^{(n)} \to I$ *in* $\mathscr{W}^1(D)$, *where* I *is a solution of the following boundary value problem with "shooting conditions":*

$$\begin{gathered}
\omega \cdot \nabla I + \beta I = s\mathscr{S}(I) + \varkappa F, \quad (\omega, x) \in D, \\
I|_{\Gamma^-} = T(I|_{\Gamma^+}), \quad (\omega, x) \in \widetilde{S}^-, \\
I|_{\Gamma_i^-} = I|_{\Gamma_j^+}, \quad (\omega, x) \in \Gamma_i^- \cap \Gamma_j^+, \quad i \neq j, \\
I|_{\Gamma^-} = J_*, \quad (\omega, x) \in \overset{*}{S}^-.
\end{gathered}$$

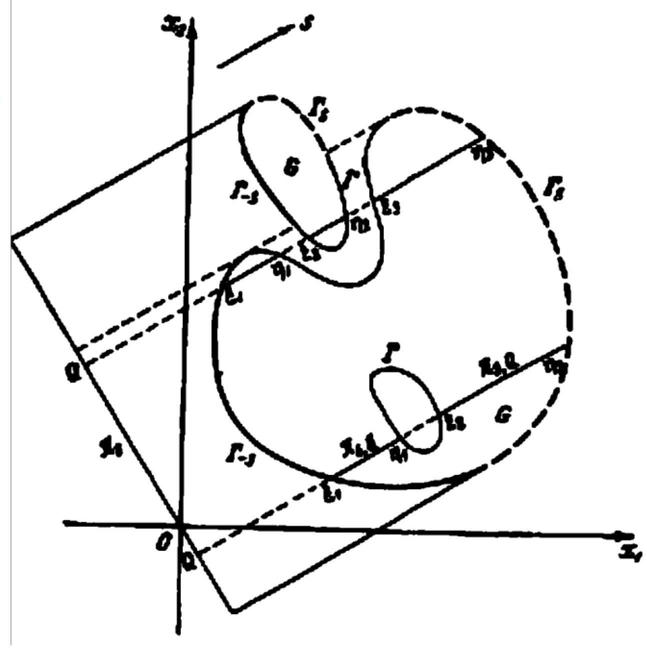
Corollary 6.1. *If additionally* $F \in L^p(D)$, $J_* \in \widehat{L}^p(\overset{*}{S}^-)$, $\{F^{(n)}\}_{n=1}^{\infty} \subset L^p(D)$, $\{J_*^{(n)}\}_{n=1}^{\infty} \subset \widehat{L}^p(\overset{*}{S}^-)$ *with some* $p \in (1, \infty]$ *and*

$$\sup_{n \geqslant 1} \|F^{(n)}\|_{L^p(D)} < \infty, \quad \sup_{n \geqslant 1} \|J_*^{(n)}\|_{\widehat{L}^p(\overset{*}{S}^-)} < \infty.$$

Then $I^{(n)} \to I$ *in* $\mathscr{W}^q(D)$ *as* $n \to \infty$ *for all* $q \in [1, p)$.

6 Некоторые свойства пространств $\mathscr{W}^p(D)$

6.1 Формулы типа Фубини



Положим

$$\widehat{\tau}^{+}(\omega,x)=\sup\{t>0 \mid x+s\,\omega\in G \quad \forall s\in(0,t)\},\quad (\omega,x)\in\Gamma^{-}, \tag{1}$$
$$\widehat{\tau}^{-}(\omega,x)=\sup\{t>0 \mid x-s\,\omega\in G \quad \forall s\in(0,t)\},\quad (\omega,x)\in\Gamma^{+}. \tag{2}$$

Теорема 6.1. *Пусть $f\in L^1(D)$. Тогда справедливы формулы*

$$\int\limits_D f(\omega,x)\,d\omega\,dx=\int\limits_{\Gamma^-}\left[\int\limits_0^{\widehat{\tau}^{+}(\omega,x)} f(\omega,x+t\omega)\,dt\right]\widehat{d\Gamma}^{-}(\omega,x), \tag{3}$$

$$\int\limits_D f(\omega,x)\,d\omega\,dx=\int\limits_{\Gamma^+}\left[\int\limits_0^{\widehat{\tau}^{-}(\omega,x)} f(\omega,x-t\omega)\,dt\right]\widehat{d\Gamma}^{+}(\omega,x). \tag{4}$$

Теорема 6.2. *Функция $w \in L^1(D)$ является обобщенной производной по направлению ω функции $f \in L^1(D)$ (то есть $w = \omega \cdot \nabla f$) тогда и только тогда, когда выполнено следующее свойство:*

Для почти всех $(\omega, x) \in \Gamma^-$ функция $f_{\omega,x}(t) = f(\omega, x+t\omega)$ принадлежит пространству $W_1^1(0, \widehat{\tau}^+(\omega, x))$, причем

$$\frac{d}{dt} f_{\omega,x}(t) = w_{\omega,x}(t) \quad \text{для почти всех} \quad t \in \big(0, \widehat{\tau}^+(\omega, x)\big),$$

где $w_{\omega,x}(t) = w(\omega, x + t\omega)$.

Следствие 6.1. *Пусть $f \in \mathscr{W}^1(D)$. Тогда:*

1. Для почти всех $(\omega, x) \in \Gamma^-$ функция $f_{\omega,x}(t) = f(\omega, x+t\omega)$ принадлежит пространству $W_1^1(0, \widehat{\tau}^+(\omega, x))$. Как следствие, для почти всех $(\omega, x) \in \Gamma^-$ существуют конечные пределы

$$\operatorname*{lim\,ap}_{t \to 0^+} f(\omega, x + t\,\omega), \qquad \operatorname*{lim\,ap}_{t \to \widehat{\tau}^+(\omega,x)} f(\omega, x + t\,\omega). \tag{5}$$

2. Для почти всех $(\omega, x) \in \Gamma^+$ функция $f_{\omega,x}(t) = f(\omega, x-t\omega)$ принадлежит пространству $W_1^1(0, \widehat{\tau}^-(\omega, x))$. Как следствие, для почти всех $(\omega, x) \in \Gamma^+$ существуют конечные пределы

6.2 Следы функций из пространств $\mathscr{W}^p(D)$

Для $f \in C^{(0,1)}(\overline{D}_j)$ следы $f|_{\Gamma_j^+}$ и $f|_{\Gamma_j^-}$ естественным образом определяются как сужения функции f на Γ_j^+ и Γ_j^-. Пусть $1 \leqslant p < \infty$ и пусть $K^\pm$ произвольные компактные подмножества множеств $\Gamma_j^\pm$ соответственно. Справедливы оценки

$$\|f|_{\Gamma_j^\pm}\|_{L^p(K^+)} \leqslant c_{K^+}\|f\|_{\mathscr{W}^p(D_j)} \quad \forall f \in C^{(0,1)}(\overline{D}_j), \tag{7}$$

с постоянными $c_{K^\pm}$, зависящими только от $K^\pm$, G_j и p.

Поскольку множество $C^{(0,1)}(\overline{D}_j)$ плотно в $\mathscr{W}^p(D_j)$, то оценки (7) позволяют расширить линейные операторы $f \to f|_{\Gamma_j^+}$ и $f \to f|_{\Gamma_j^-}$ до линейных непрерывных операторов, действующих из $\mathscr{W}^p(D_j)$ в $L^p(K^+)$ и $L^p(K^-)$ соответственно и удовлетворяющих оценкам

$$\|f|_{\Gamma_j^\pm}\|_{L^p(K^\pm)} \leqslant c_{K^\pm}\|f\|_{\mathscr{W}^p(D_j)} \quad \forall f \in \mathscr{W}^p(D_j),$$

Поскольку множества Γ_j^+ и Γ_j^- могут быть представлены в виде счетных объединений расширяющихся компактных множеств, то для функции $f \in \mathscr{W}^p(D_j)$, $1 \leqslant p < \infty$ определены следы $f|_{\Gamma_j^+} \in L^p_{loc}(\Gamma_j^+)$ и $f|_{\Gamma_j^-} \in L^p_{loc}(\Gamma_j^-)$. Как следствие, для всякой функции $f \in \mathscr{W}^p(D)$, $1 \leqslant p < \infty$ определены следы $f|_{\Gamma^+} \in L^p_{loc}(\Gamma^+)$ и $f|_{\Gamma^-} \in L^p_{loc}(\Gamma^-)$.

Известно, что следы $f|_{\Gamma^\pm}$ функции $f \in \mathscr{W}^p(D)$ при $1 \leqslant p < \infty$ не обязаны принадлежать пространствам $L^p(\Gamma^\pm)$ и даже пространствам $\widehat{L}^p(\Gamma^\pm)$.

Точные результаты о свойствах следов функций из пространств $\mathscr{W}^p(D)$ с $1 \leqslant p < \infty$ были получены в работах Cessenat M. (1985, 1986 г.г.) и независимо в работе Агошкова В.И. (1987 г.).

Показано, что операторы следов $f \to f|_{\Gamma^+}$ и $f \to f|_{\Gamma^+}$ являются линейными непрерывными операторами из пространства $\mathscr{W}^p(D)$ в весовые пространства $\widehat{L}^p_{\widehat{\tau}^-}(\Gamma^+)$ и $\widehat{L}^p_{\widehat{\tau}^+}(\Gamma^-)$ соответственно, где $\widehat{\tau}^-$, $\widehat{\tau}^+$ определены формулами (1), (2).

Справедлива следующая теорема.

Теорема 6.3. *Пусть* $f \in \mathscr{W}^p(D_j)$, $1 \leqslant p \leqslant \infty$. *Тогда для следов* $f|_{\Gamma^-}$ *и* $f|_{\Gamma^+}$ *справедливы формулы*

$$f|_{\Gamma^-}(\omega, x) = \lim_{t \to 0^+} \text{ap}\ f(\omega, x + t\,\omega) \quad \textit{для почти всех} \quad (\omega, x) \in \Gamma_j^-, \qquad (8)$$

$$f|_{\Gamma^+}(\omega, x) = \lim_{t \to 0^+} \text{ap}\ f(\omega, x - t\,\omega) \quad \textit{для почти всех} \quad (\omega, x) \in \Gamma_j^+. \qquad (9)$$