Group analysis of systems of two second-order ordinary differential equations

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Abrao-Durso
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Outline

1. Method of the study
   a) admitted Lie group,
   b) invariant solutions,
   c) applications to ODEs (single equation, systems)
2. Linear systems of two second-order ODEs
   a) with constant coefficients,
   b) with variable coefficients,
3. Linear systems of more then two second-order ODEs
4. Discussion

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Symmetries of Integro-Differential Equations

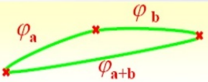
Invertible transformations:

$$\overline{z}^i = \varphi^i(z\,;a),\quad i = 1,\dots,N$$

$z = (x,u)$, $\varphi = (f,g) \in R^N$, $N = n+m$
$x = (x_1,\dots,x_n) \in R^n$: Ind. var.
$u = (u^1,\dots,u^m) \in R^m$: Dep. var.
a : Parameter in symmetrical interval $\Delta \subset R$

Local one-parameter Lie group G

$$(z) \xrightarrow{\varphi_a} (\overline{z})$$



1. $\varphi(z\,;0) = z \quad \forall\ z \in V$
2. $\varphi(\varphi(z\,;a);\,b) = \varphi(z\,;a+b)\ \forall\ z \in V,\ a,b,a+b \in \Delta$
3. $a \in \Delta,\ \varphi(z\,;a) = z\ \forall\ z \in V \ \Rightarrow\ a = 0$
4. $\varphi \in C^{\infty}(V,\Delta)$, V is open subset of $R^n \times R^m$

Group Classification
Lie Group of Transformations

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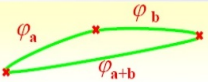

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Group Classification

Infinitesimal Generators

Taylor series expansion :

$$\bar{x}_i \approx x_i + \xi^{x_i}(x,u)a$$

$$\bar{u}^j \approx u^j + \zeta^{u^j}(x,u)a$$

$$\xi^{x_i}(x,u) = \left.\frac{\partial f^i(x,u;a)}{\partial a}\right|_{a=0}$$

$$\zeta^{u^j}(x,u) = \left.\frac{\partial g^j(x,u;a)}{\partial a}\right|_{a=0}$$

$$X = \xi^{x_i}(x,u)\partial_{x_i} + \zeta^{u^j}(x,u)\partial_{u^j}$$

is infinitesimal generator

Th$^{\mathrm{m}}$ (Lie) Lie group (G) determined by the solution of

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$\{\varphi_a\}$ ⇄ (Lie Equations) $X = \xi^{x_i}(x,u)\partial_{x_i} + \zeta^{u^j}(x,u)\partial_{u^j}$

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$$S(x, u, p) = 0$$

1. Find admitted Lie algebra.

2. Choose a subalgebra.

3. Find invariants of the subalgebra.

4. Construct a representation of an invariant or partially invariant solution.

5. Substitute a representation of the solution into the original system of differential equations.

6. Make a compatibility analysis of the reduced system

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2. To classify equations with arbitrary elements (group classification)

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Computational Procedure

1. Equivalence Transformations
2. Partial solving of the Determining Equations
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1. Linear change of the dependent variables

$$\widetilde{\mathbf{y}} = P\mathbf{y}$$

2. the change

$$\widetilde{\mathbf{y}} = \mathbf{y} + \mathbf{g}(x)$$

3. the change

$$\widetilde{x} = \varphi(x), \ \widetilde{\mathbf{y}} = \mathbf{y}\psi(x),$$

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Determining equations ($\mathbf{y} = (y, z)$)

$$F_z(\xi' z + zk_4 + yk_3 + \zeta_2) + F_y(\xi' y + zk_2 + yk_1 + \zeta_1)$$
$$+2F_x\xi - \xi''' y + 3\xi' F - \zeta_1'' - k_1 F - k_2 G = 0,$$

$$G_z(\xi' z + zk_4 + yk_3 + \zeta_2) + G_y(\xi' y + zk_2 + yk_1 + \zeta_1)$$
$$+2G_x\xi - \xi''' z + 3\xi' G - \zeta_2'' - k_3 F - k_4 G = 0,$$

where an admitted generator has the form

$$X = 2\xi(x)\partial_x + (y\xi'(x) + k_1 y + k_2 z + \zeta_1(x))\partial_y + (\xi' z + k_3 y + k_4 z + \zeta_2(x))\partial_z,$$

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The equivalence transformations

$$x_1 = \alpha(x), \;\; y_1 = y\beta(x), \;\; z_1 = z\beta(x),$$

where

$$\alpha''\beta = 2\alpha'\beta', \quad (\alpha'\beta \neq 0), \;\; 2\xi\beta'/\beta + \xi' = 0$$

reduces the generator X_o to

$$X_o = \partial_x + (a_{11}y + a_{12}z)\partial_y + (a_{21}y + a_{22}z)\partial_z.$$

The determining equations become

$$\left((A\mathbf{y})^t\nabla\right)\mathbf{F} + \mathbf{F}_x - A\mathbf{F} = 0, \tag{1}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; \nabla = \begin{pmatrix} \partial_y \\ \partial_z \end{pmatrix}.$$

Simplifications of a generator ($\mathbf{y} = (y, z)$)

Case $\xi \neq 0$

The equivalence transformations

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The change $\widetilde{\mathbf{y}} = P\mathbf{y}$ gives

$$P^{-1}\left(\left((\widetilde{A}\widetilde{\mathbf{y}})^t\widetilde{\nabla}\right)\widetilde{\mathbf{F}} + \widetilde{\mathbf{F}}_x - \widetilde{A}\widetilde{\mathbf{F}}\right) = 0,$$

where

$$\widetilde{A} = PAP^{-1}, \quad \widetilde{\mathbf{F}}(x, \widetilde{\mathbf{y}}) = P\mathbf{F}(x, P^{-1}\widetilde{\mathbf{y}}).$$

$$J_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad J_2 = \begin{pmatrix} a & c \\ -c & a \end{pmatrix}, \quad J_3 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \qquad (2)$$

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Simplifications of a generator ($\mathbf{y} = (y, z)$)

Case $\xi \neq 0$ and $A = J_1$

$$ayF_y + bzF_z + F_x = aF,$$
$$ayG_y + bzG_z + G_x = bG.$$

The general solution of these equations is

$$F(x, u, v) = e^{ax} f(u, v), \quad G(x, u, v) = e^{bx} g(u, v)$$

$$u = ye^{-ax}, \quad v = ze^{-bx}.$$

$$X_o = \partial_x + ay\partial_y + bz\partial_z.$$

Simplifications of a generator ($\mathbf{y} = (y, z)$)

Case $\xi \neq 0$ and $A = J_2$

$$F(x,u,v) = e^{ax}\left(\cos(cx)f(u,v) + \sin(cx)g(u,v)\right),$$
$$G(x,y,z) = e^{ax}\left(-\sin(cx)f(u,v) + \cos(cx)g(u,v)\right)$$

$$u = e^{-ax}(y\cos(cx) - z\sin(cx)), \quad v = e^{-ax}(y\sin(cx) + z\cos(cx)),$$

$$X_o = \partial_x + (ay + cz)\partial_y + (-cy + az)\partial_z.$$

Simplifications of a generator ($\mathbf{y} = (y, z)$)

Case $\xi \neq 0$ and $A = J_3$

$$F(x,u,v) = e^{ax}\left(f(u,v) + xg(u,v)\right), \quad G(x,y,z) = e^{ax}g(u,v),$$

$$u = e^{-ax}(y - zx), \quad v = e^{-ax}z$$

$$X_o = \partial_x + (ay + z)\partial_y + az\partial_z.$$

Linear system of equations with constant coefficients
Simplifications

$$\mathbf{y}'' = A(x)\mathbf{y}' + B(x)\mathbf{y},$$

$$\mathbf{y} = C(x)\mathbf{y}_1,$$

$$\mathbf{y}_1'' = \bar{A}\mathbf{y}_1' + \bar{B}\mathbf{y}_1,$$

$$\bar{A} = C^{-1}(AC - 2C'), \quad \bar{B} = C^{-1}(BC + AC' - C'').$$

$$C' = \frac{1}{2}AC,$$

$$\frac{d}{dx}\bar{B} = C^{-1}(BA - AB)C. \quad \Leftrightarrow \quad BA = AB$$

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Theorem. A linear system with non-commuting constant matrices A and B admits a nontrivial symmetry if this system is equivalent to a linear system with the matrices A and B of the form

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} b_{22}+4 & b_{12} \\ 0 & b_{22} \end{pmatrix}, \quad (b_{12} \neq 0). \qquad (3)$$

The admitting symmetries (except generic) of the system with matrices (3) are

$$\begin{array}{llll} \text{if} & b_{22} \neq -15/4 & : & X_1 = e^{-2x} z\partial_y; \\ \text{if} & b_{22} = -15/4 & : & X_1 = e^{-2x} z\partial_y, \\ & & & X_2 = e^{-x}\left(2\partial_x - y\partial_y + 3z\partial_z\right). \end{array}$$

Linear system of equations with nonconstant coefficients

$$\begin{cases} F = \alpha_{11} y + e^{x} z, \\ G = e^{-x}\alpha_{21} y + \alpha_{22} z, \end{cases} \implies \partial_x - z\partial_z$$

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Linear system of equations. Algebraic approach

Simplifications of the linear system $\mathbf{y}'' = A(x)\mathbf{y}$

$$\tilde{x} = \varphi(x), \quad \tilde{\mathbf{y}} = \psi(x)\mathbf{y}$$

$$\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi},$$

$$\mathbf{y}'' = A\mathbf{y} \Rightarrow \tilde{\mathbf{y}}'' = \tilde{A}\tilde{\mathbf{y}}$$

$$\tilde{A} = \varphi'^{-2}\left(A - \frac{\rho''}{\rho}E\right), \quad \rho = \frac{1}{\psi}.$$

$$trace(\tilde{A}) = 0$$

Wafo Soh & Mahomed (2000)

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Linear system of equations. Algebraic approach

Determining equations

$$A = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix},$$

$$X = 2\xi(x)\partial_x + (y\xi'(x) + q_1 y + q_2 z)\partial_y + (\xi' z + q_3 y + q_4 z)\partial_z$$

$$\begin{gathered} 2a'\xi + 4a\xi' + bq_3 - cq_1 = 0, \\ 2b'\xi + 2aq_1 + b(4\xi' + q_4 - q_2) = 0, \\ 2c'\xi - 2aq_3 + c(4\xi' - q_4 + q_2) = 0. \end{gathered}$$

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	X_1	X_2	X_3
X_1	0	$-2X_1$	$-X_2$
X_2	$2X_1$	0	$-2X_3$
X_3	X_2	$2X_3$	0

	X_4	X_5	X_6
X_4	0	X_6	$-2X_4$
X_5	$-X_6$	0	$2X_5$
X_6	$2X_4$	$-2X_5$	$0.$

1.1.	$X_2 + \gamma(X_4 - X_5)$	3.1.	$X_1 \pm X_3 + \gamma(X_4 - X_5)$
1.2.	$X_2 + \gamma X_5$	3.2.	$X_1 \pm X_3 + \gamma X_5$
1.3.	$X_2 + \gamma X_6$	3.3.	$X_1 \pm X_3 + \gamma X_6$
1.4.	X_2	3.4.	$X_1 \pm X_3$
2.1.	$X_3 + \gamma(X_4 - X_5)$	4.1.	$X_4 - X_5$
2.2.	$X_3 + \gamma X_5$	4.2.	X_5
2.3.	$X_3 + \gamma X_6$	4.3.	X_6
2.4.	X_3		

	X_1	X_2	X_3
X_1	0	$-2X_1$	$-X_2$
X_2	$2X_1$	0	$-2X_3$
X_3	X_2	$2X_3$	0

	X_4	X_5	X_6
X_4	0	X_6	$-2X_4$
X_5	$-X_6$	0	$2X_5$
X_6	$2X_4$	$-2X_5$	$0.$

1.1.	$X_2+\gamma(X_4-X_5)$	3.1.	$X_1\pm X_3+\gamma(X_4-X_5)$
1.2.	$X_2+\gamma X_5$	3.2.	$X_1\pm X_3+\gamma X_5$
1.3.	$X_2+\gamma X_6$	3.3.	$X_1\pm X_3+\gamma X_6$
1.4.	X_2	3.4.	$X_1\pm X_3$
2.1.	$X_3+\gamma(X_4-X_5)$	4.1.	X_4-X_5
2.2.	$X_3+\gamma X_5$	4.2.	X_5
2.3.	$X_3+\gamma X_6$	4.3.	X_6
2.4.	X_3		

	X_1	X_2	X_3
X_1	0	$-2X_1$	$-X_2$
X_2	$2X_1$	0	$-2X_3$
X_3	X_2	$2X_3$	0

	X_4	X_5	X_6
X_4	0	X_6	$-2X_4$
X_5	$-X_6$	0	$2X_5$
X_6	$2X_4$	$-2X_5$	$0.$

1.1.	$X_2 + \gamma(X_4 - X_5)$	3.1.	$X_1 \pm X_3 + \gamma(X_4 - X_5)$
1.2.	$X_2 + \gamma X_5$	3.2.	$X_1 \pm X_3 + \gamma X_5$
1.3.	$X_2 + \gamma X_6$	3.3.	$X_1 \pm X_3 + \gamma X_6$
1.4.	X_2	3.4.	$X_1 \pm X_3$
2.1.	$X_3 + \gamma(X_4 - X_5)$	4.1.	$X_4 - X_5$
2.2.	$X_3 + \gamma X_5$	4.2.	X_5
2.3.	$X_3 + \gamma X_6$	4.3.	X_6
2.4.	X_3		

Example $X_2 + \gamma(X_4 - X_5)$

$$\begin{gathered} 2xa' + 4a + \gamma(b+c) = 0, \\ xb' + 2b - \gamma a = 0, \\ xc' + 2c - \gamma a = 0. \end{gathered}$$

$$\begin{aligned} a &= \frac{C_1 \sin(\gamma \ln x) + C_2 \cos(\gamma \ln x)}{x^2}, \\ b &= \frac{k - 2C_1 \cos(\gamma \ln x) + 2C_2 \sin(\gamma \ln x)}{2x^2}, \\ c &= \frac{-k - 2C_1 \cos(\gamma \ln x) + 2C_2 \sin(\gamma \ln x)}{2x^2}. \end{aligned}$$

$$\begin{gathered} F = y(\sin(x) + c_2) + z(\cos(x) - c_1), \\ G = y(\cos(x) + c_1) + z(-\sin(x) + c_2), \end{gathered}$$

$$2\partial_x + z\partial_y - y\partial_z$$

Example $X_2 + \gamma(X_4 - X_5)$

$$\begin{gathered} 2xa' + 4a + \gamma(b + c) = 0, \\ xb' + 2b - \gamma a = 0, \\ xc' + 2c - \gamma a = 0. \end{gathered}$$

$$a = \frac{C_1 \sin(\gamma \ln x) + C_2 \cos(\gamma \ln x)}{x^2},$$

$$b = \frac{k - 2C_1 \cos(\gamma \ln x) + 2C_2 \sin(\gamma \ln x)}{2x^2},$$

$$c = \frac{-k - 2C_1 \cos(\gamma \ln x) + 2C_2 \sin(\gamma \ln x)}{2x^2}.$$

$$\begin{gathered} F = y(\sin(x) + c_2) + z(\cos(x) - c_1), \\ G = y(\cos(x) + c_1) + z(-\sin(x) + c_2), \end{gathered}$$

$$2\partial_x + z\partial_y - y\partial_z$$

Linear system of equations. Algebraic approach

Solutions of the determining equations

Example $X_2 + \gamma(X_4 - X_5)$

$$\begin{gathered} 2xa' + 4a + \gamma(b + c) = 0, \\ xb' + 2b - \gamma a = 0, \\ xc' + 2c - \gamma a = 0. \end{gathered}$$

$$\begin{aligned} a &= \frac{C_1 \sin(\gamma \ln x) + C_2 \cos(\gamma \ln x)}{x^2}, \\ b &= \frac{k - 2C_1 \cos(\gamma \ln x) + 2C_2 \sin(\gamma \ln x)}{2x^2}, \\ c &= \frac{-k - 2C_1 \cos(\gamma \ln x) + 2C_2 \sin(\gamma \ln x)}{2x^2}. \end{aligned}$$

$$\begin{gathered} F = y(\sin(x) + c_2) + z(\cos(x) - c_1), \\ G = y(\cos(x) + c_1) + z(-\sin(x) + c_2), \end{gathered}$$

$$2\partial_x + z\partial_y - y\partial_z$$

Example $X_2 + \gamma(X_4 - X_5)$

$$\begin{gathered} 2xa' + 4a + \gamma(b + c) = 0, \\ xb' + 2b - \gamma a = 0, \\ xc' + 2c - \gamma a = 0. \end{gathered}$$

$$\begin{aligned} a &= \frac{C_1 \sin(\gamma \ln x) + C_2 \cos(\gamma \ln x)}{x^2}, \\ b &= \frac{k - 2C_1 \cos(\gamma \ln x) + 2C_2 \sin(\gamma \ln x)}{2x^2}, \\ c &= \frac{-k - 2C_1 \cos(\gamma \ln x) + 2C_2 \sin(\gamma \ln x)}{2x^2}. \end{aligned}$$

$$\begin{aligned} F &= y(\sin(x) + c_2) + z(\cos(x) - c_1), \\ G &= y(\cos(x) + c_1) + z(-\sin(x) + c_2), \end{aligned}$$

$$2\partial_x + z\partial_y - y\partial_z$$

Example $X_2 + \gamma(X_4 - X_5)$

$$\begin{gathered} 2xa' + 4a + \gamma(b + c) = 0, \\ xb' + 2b - \gamma a = 0, \\ xc' + 2c - \gamma a = 0. \end{gathered}$$

$$\begin{aligned} a &= \frac{C_1 \sin(\gamma \ln x) + C_2 \cos(\gamma \ln x)}{x^2}, \\ b &= \frac{k - 2C_1 \cos(\gamma \ln x) + 2C_2 \sin(\gamma \ln x)}{2x^2}, \\ c &= \frac{-k - 2C_1 \cos(\gamma \ln x) + 2C_2 \sin(\gamma \ln x)}{2x^2}. \end{aligned}$$

$$\begin{aligned} F &= y(\sin(x) + c_2) + z(\cos(x) - c_1), \\ G &= y(\cos(x) + c_1) + z(-\sin(x) + c_2), \end{aligned}$$

$$2\partial_x + z\partial_y - y\partial_z$$

$$X_2 + x_4X_4 + x_5X_5 + x_6X_6,$$
$$X_3 + k(y_4X_4 + y_5X_5 + y_6X_6),$$

$$k = 0 \Longrightarrow A = const$$

$$X_2 + x_4X_4 + x_5X_5 + x_6X_6,$$
$$X_3 + k(y_4X_4 + y_5X_5 + y_6X_6),$$

$$k = 0 \Longrightarrow A = const$$

$$X_2 + x_4 X_4 + x_5 X_5 + x_6 X_6,$$
$$X_3 + k(y_4 X_4 + y_5 X_5 + y_6 X_6),$$

$$k = 0 \implies A = const$$

$$X_2 + x_4 X_4 + x_5 X_5 + x_6 X_6,$$
$$X_3 + k(y_4 X_4 + y_5 X_5 + y_6 X_6),$$

$$k = 0 \Longrightarrow A = const$$

Linear system of three equations

$$\begin{pmatrix} y'' \\ z'' \\ u'' \end{pmatrix} = \begin{pmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{pmatrix} \begin{pmatrix} y \\ z \\ u \end{pmatrix}$$

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