Application of the method of differential constraints to constructing exact solutions of the gas dynamics equations

S.V. Meleshko

School of Mathematics, Institute of Science, Suranaree University of Technology, Nakhon Ratchasima, Thailand

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   - Group classification
2. Conservation law
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Equations describing motion of fluids with internal inertia

S.L.Gavrilyuk, V.M.Teshukov (2001)

$$\dot{\rho} + \rho div(u) = 0, \quad \rho\dot{u} + \nabla p = 0, \quad \dot{S} = 0$$

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$$p = \rho\frac{\delta W}{\delta \rho} - W, \quad W = W(\rho, \dot{\rho}, S)$$

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$$p = \rho\frac{\delta W}{\delta \rho} - W, \quad W = W(\rho, \dot{\rho}, S)$$

$$p = \rho\left(W_\rho - (W_{\dot{\rho}} - div\left(uW_{\dot{\rho}}\right)\right) - W,$$

t is time,

∇ is the gradient operator with respect to space variables,

ρ is the fluid density,

u is the velocity field,

$W(\rho, \dot{\rho}, S)$ is a given potential,

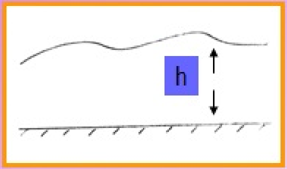
'dot' denotes the material time derivative: $\dot{f} = \dfrac{df}{dt} = f_t + u\nabla f$,

$\dfrac{\delta W}{\delta \rho}$ denotes the variational derivative of W with respect to ρ at a fixed value of u.

Example 1: Iordanski-Kogarko-Wingaarden model (1960, 1961, 1968)

$$W = \psi(\rho) - k\dot{\rho}^2 \frac{\rho^{8/3}}{(\alpha - \rho)^{1/3}}$$

Example 2: Green-Naghdi equations (1975)



$$\begin{cases} h_t + \mathrm{div}(hu) = 0 \\ h(u_t + (u,\nabla)u) + gh\nabla h + \dfrac{\varepsilon^2}{3}\nabla(h^2\ddot{h}) = 0 \end{cases}$$

$$h = \rho$$

$$W = \frac{g\rho^2}{2} - \frac{\varepsilon^2 \rho\dot{\rho}^2}{6}$$

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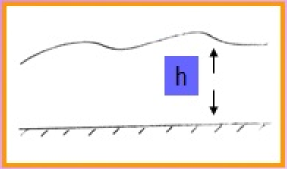

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Yu.Yu.Bagderina, A.P.Chupakhin (2005)
Invariant and partially invariant solutions of the Green–Naghdi equations

Isentropic flows

	$W(\rho,\dot\rho)$	Extensions	Remarks
M_1	$-\alpha\dot\rho^2\rho^{-3}+\beta\rho^3$	$X_2,\ X_1-X_3-2X_6$	$\alpha\beta\neq0$
M_2	$-\alpha\dot\rho^2\rho^{-3}$	$X_2,\ X_1-X_3,\ X_6$	$\alpha\neq0$
M_3	$\varphi(\rho)\dot\rho^p$	$(p-2)X_3-2X_6$	$p(p-1)\neq0$
M_4	$(\alpha\rho+\gamma)\ln(\dot\rho)+\varphi_2(\rho)$	X_3+X_6	φ_2 arbitrary
M_5	$\varphi(\rho)\dot\rho\ln(\dot\rho)$	X_3+2X_6	
M_6	$\rho^{\lambda-2k}\varphi(\dot\rho\rho^k)+\varphi_2(\rho)$	$2X_1+(\lambda+1)X_3+2(k+1)X_6$	$\varphi_2''=C_2\rho^{\lambda-2(k+1)}$
M_7	$\alpha\rho^\lambda\dot\rho^p+\varphi_2(\rho)$	$p(2X_1+(\lambda+p-1)X_3)$ $-(\lambda+p-\mu-2)((p-2)X_3-2X_6)$	$\varphi_2''=C_2\rho^\mu\neq0$ $p(p-1)\neq0$
M_8	$\alpha\rho^\lambda\dot\rho^p$	$2X_1+(\lambda+p-1)X_3,\ (p-2)X_3-2X_6$	$p(p-1)\neq0$
M_9	$\alpha\rho^\lambda\ln(\dot\rho)+\varphi_2(\rho)$	X_3+X_6	φ_2 arbitrary $\lambda(\lambda-1)=0$
M_{10}		$X_3+X_6,\ 2X_1+(\lambda-1)X_3$	$\varphi_2''=C_2\rho^{\lambda-2}$ $\lambda(\lambda-1)=0$
M_{11}		$X_1+\frac{\lambda-1}{2}X_3+X_6+\frac{\alpha}{C\lambda(\lambda-1)}(X_3+X_6)$	$\varphi_2''=\rho^{\lambda-2}(\alpha\ln(\rho)+\beta)$ $\lambda(\lambda-1)\neq0$
M_{12}	$\alpha\rho^\lambda\dot\rho\ln(\dot\rho)+\varphi_2(\rho)$	$2X_1+\lambda X_3+(\lambda-\mu-1)(X_3+2X_6)$	$\varphi_2''=C_2\rho^\mu\neq0$
M_{13}	$\alpha\rho^\lambda\dot\rho\ln(\dot\rho)$	$2X_1+\lambda X_3,\ X_3+2X_6$	
M_{14}	$\alpha\dot\rho^2+\varphi_2(\rho)$	$2X_1+X_3-\mu X_6$	$\varphi_2''=C_2\rho^\mu\neq0$
M_{15}	$\alpha\dot\rho^2$	$2X_1+X_3,\ X_6$	

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M_4	$(\alpha\rho+\gamma)\ln(\dot\rho)+\varphi_2(\rho)$	X_3+X_6	φ_2 arbitrary
M_5	$\varphi(\rho)\dot\rho\ln(\dot\rho)$	X_3+2X_6	
M_6	$\rho^{\lambda-2k}\varphi(\dot\rho\rho^k)+\varphi_2(\rho)$	$2X_1+(\lambda+1)X_3+2(k+1)X_6$	$\varphi_2''=C_2\rho^{\lambda-2(k+1)}$
M_7	$\alpha\rho^\lambda\dot\rho^p+\varphi_2(\rho)$	$p(2X_1+(\lambda+p-1)X_3)$ $-(\lambda+p-\mu-2)((p-2)X_3-2X_6)$	$\varphi_2''=C_2\rho^\mu\neq 0$ $p(p-1)\neq 0$
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M_{12}	$\alpha\rho^\lambda\dot\rho\ln(\dot\rho)+\varphi_2(\rho)$	$2X_1+\lambda X_3+(\lambda-\mu-1)(X_3+2X_6)$	$\varphi_2''=C_2\rho^\mu\neq 0$
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M_{14}	$\alpha\dot\rho^2+\varphi_2(\rho)$	$2X_1+X_3-\mu X_6$	$\varphi_2''=C_2\rho^\mu\neq 0$
M_{15}	$\alpha\dot\rho^2$	$2X_1+X_3,\ X_6$	

A.Hematulin, S.V.Meleshko and S.L.Gavrilyuk (2007)

Nonisentropic flows

	$W(\rho, \dot\rho, S)$	Extensions	Remarks
M_1	$q_0\rho^{-3}\dot\rho^2 + \rho^3 S$	$X_p,\ \ X_4 - 2S\partial_S,\ \ X_5 - X_6$	
M_2	$\rho^{-3}\dot\rho^2 S + q_1\rho^3 S^k$	$X_p,\ \ X_5 - X_6,\ \ X_6 - (k+1)X_4 + 2S\partial_S$	$q_1 \neq 0$
M_3	$\rho^{-3}\dot\rho^2 S + \rho^3\mu(S)$	$X_p,\ \ X_5 - X_6$	$\mu' \neq q_1 S^k, \mu \neq 0$
M_4	$\rho^{-3}\dot\rho^2 S$	$X_p,\ \ X_4,\ \ X_5 - X_6,\ \ X_6 + 2S\partial_S$	
M_5	$\rho^\alpha\phi(\dot\rho\rho^\beta, S)$	$-(\alpha+\beta)X_4 + (\beta + \frac{\alpha+1}{2})X_5 + X_6$	$\alpha(\alpha-1) \neq 0$
M_6	$\phi(\dot\rho\rho^{-\gamma}, S) - q_0\ln(\rho)$	$2\gamma(X_4 - X_5) + X_5 + 2X_6$	
M_7	$\rho\phi(\dot\rho\rho^\alpha, S) + \rho\ln(\rho)\psi(S)$	$-(\alpha+1)(X_4 - X_5) + X_6$	
M_8	$\dot\rho\ln(\dot\rho)\phi(\rho, S)$	X_5	
M_9	$\dot\rho^\alpha\phi(\rho, S)$	$X_4 + \frac{2-\alpha}{2(\alpha-1)}X_5$	$\alpha(\alpha-1) \neq 0$
M_{10}	$\phi(\rho, S) + \ln(\dot\rho)(q_0 + \rho\psi(S))$	$-X_4 + X_5$	$q_0\psi \neq 0$
M_{11}	$e^{\alpha S}\phi(\rho e^{-S}, \dot\rho e^{\beta S})$	$-(\alpha+\beta)X_4 + (\beta + \frac{\alpha+1}{2})X_5 + X_6 + \partial_S$	$\alpha \neq 0$
M_{12}	$\phi(\rho e^{-S}, \dot\rho e^{-\gamma S}) + q_0 S$	$2\gamma(X_4 - X_5) + X_5 + 2(X_6 + \partial_S)$	
M_{13}	$e^{\alpha S}\phi(\rho, \dot\rho e^{(1-\frac{\alpha}{2})S})$	$(-\alpha-2)X_4 + 2X_5 + 2\partial_S$	$\alpha \neq 0$
M_{14}	$\phi(\rho, \dot\rho e^S) + q_0 S$	$-X_4 + X_5 + \partial_S$	
M_{15}	$e^{-2S}\phi(\rho, \dot\rho e^S)$	$X_4 + \partial_S$	
M_{16}	$h(\rho)\dot\rho^{-2} + \alpha S$	$\partial_S,\ \ -3X_4 + 2X_5 + 2S\partial_S$	$h\alpha \neq 0$

P.Siriwat, C.Kaewmanee and S. V. Meleshko (2015)

Group classification of one-dimensional nonsentropic equations of fluids with internal inertia II. General case

Nonisentropic flows

	$W(\rho,\dot\rho,S)$	Extensions	Remarks
M_{17}	$\rho^{2q_2-2\alpha-1}e^{-2(\beta-q_1)S}\phi\left(\dot\rho\rho^{\alpha}e^{\beta S}\right)$	$(\beta-2q_1)X_4+q_1X_5+\partial_S,\ (\alpha+1-2q_2)$ $X_4+q_2X_5+X_6$	$(\alpha-q_2+1)(2(\alpha-q_2)+1)\neq 0$
M_{18}	$\rho e^{\alpha S}\left(q_0\ln(\rho)+\phi\left(\dot\rho\rho^{\lambda}e^{\beta S}\right)\right)$	$-(\alpha+\beta)X_4+(\frac{\alpha}{2}+\beta)X_5+\partial_S,\ -(\lambda+1)$ $(X_4-X_5)+X_6$	
M_{19}	$q_0\ln(\rho)+\alpha S+\phi\left(\dot\rho\rho^{q_2-1/2}e^{q_1S}\right)$	$\gamma_1(X_4-X_5)+\partial_S,\ 2\gamma_2(X_4-X_5)$ $+X_5+2X_6$	
M_{20}	$e^{-2(\beta-q_1)S}\phi\left(\dot\rho\rho^{\alpha}e^{\beta S}\right).$	$(\beta-2q_1)X_4+q_1X_5+\partial_S,\ 2\gamma_2(X_4-X_5)$ $+X_5+2X_6$	$\beta-q_1\neq 0$
M_{21}	$\dot\rho^{\alpha}e^{\beta S}\phi\left(\rho e^{-\beta_1S}\right)$	$(\beta-(1-\alpha)\beta_1)X_4+(\alpha-2)\beta_1X_6$ $+(\alpha-2)\partial_S,\ (2-2\alpha)X_4+(\alpha-2)X_5$	$(\alpha-1)(\alpha-2)\neq 0$
M_{22}	$\dot\rho\ln(\dot\rho)e^{-\gamma_1S}\phi\left(\rho e^{-\beta_1S}\right)$	$\gamma_1X_4+\beta_1X_6+\partial_S,\ X_5$	
M_{23}	$\dot\rho^2e^{\alpha S}\phi\left(\rho e^{\beta S}\right)$	$(\alpha-\beta)X_5-2\beta X_6+2\partial_S,\ X_4$	
M_{24}	$\rho^{-\gamma_2}\dot\rho\ln(\dot\rho)\phi(S)$	$X_5,\ \gamma_2X_4+X_6$	
M_{25}	$\rho^{\frac{2q_2-1}{2q_1+1}}\dot\rho^{\frac{2(q_1+1)}{2q_1+1}}\phi(S)$	$X_4+q_1X_5,\ q_2X_5+X_6$	$(1+2q_1)(1+q_1)\neq 0,$
M_{26}	$\rho\left(\ln(\rho)\psi(S)+\ln(\dot\rho)\phi(S)\right)$	$X_4-X_5,\ X_6$	
M_{27}	$q_0\ln(\rho)+\alpha\ln(\dot\rho)+S$	$X_4-X_5,\ X_5+2X_6$	
M_{28}	$q_0\ln(\dot\rho)+\alpha\ln(\rho)+S$	$\partial_S,\ X_4+2X_6,\ -X_4+X_5$	
M_{29}	$\rho e^{-S}\left(\ln(\dot\rho)+q_0\ln(\rho)\right)$	$X_4+2\partial_S,\ X_6,\ -X_4+X_5$	
M_{30}	$q_0\rho^{-q_2}\dot\rho\ln(\dot\rho)e^{-q_1S}$	$q_1X_4+\partial_S,\ q_2X_4+X_6,\ X_5$	
M_{31}	$\rho^{\alpha}\dot\rho^{\beta}e^{\lambda S}$	$-\lambda X_4+(2-\beta)\partial_S,\ (1-(\beta+\alpha))X_4$ $+(2-\beta)X_6,\ (2\beta-2)X_4+(2-\beta)X_5$	$\frac{2\beta-2}{(2-\beta)^2}\neq 0$
M_{32}	$q_0\dot\rho^2\rho^{\alpha}e^{2\beta S}$	$\beta X_5+\partial_S,\ (\alpha+1)X_5+2X_6,\ X_4$	
M_{33}	$q_0\rho^{-\gamma_3}\dot\rho\ln(\dot\rho)-S$	$\partial_S,\ q_2X_5+S\partial_S,\ \gamma_3X_4+q_3X_5+X_6$	$\alpha q_0\neq 0$
M_{34}	$q_0\rho^{\alpha}\dot\rho^{\beta}\ln(\dot\rho)-S$	$\partial_S,\ (1-\beta)X_4+\beta\beta_2X_6+\beta S\partial_S,$ $-\alpha X_4+\beta X_5$	$\beta(1-\beta)\neq 0$
M_{35}	$q_0\rho\ln(\dot\rho/\rho)-\alpha S$	$\partial_S,\ q_2X_5+X_6+S\partial_S,\ X_4$	$\gamma_3+2\neq 0$

P.Siriwat, C.Kaewmanee and S. V. Meleshko (2015)

Group classification of one-dimensional nonisentropic equations of fluids with internal inertia II. General case

Noether's theorem

$$XF + FD_i\xi^i = W^k \frac{\delta F}{\delta u^k} + D_i(\mathcal{N}^i F)$$

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$$\mathcal{N}^i F = \xi^i F + W^k \frac{\delta F}{\delta u_i^k} + \sum_{s=1} D_{i_1} \ldots D_{i_s}(W^k) \frac{\delta F}{\delta u^k_{i i_1 i_2 \ldots i_s}}, \quad (i = 1, 2, \ldots, n).$$

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$$W^k = \eta^k - \xi^i u_i^k, \quad (k = 1, 2, ..., m),$$

$$\frac{\delta}{\delta u^k} = \frac{\partial}{\partial u^k} + \sum_{s=1} (-1)^s D_{i_1}...D_{i_s} \frac{\partial}{\partial u^k_{i_1i_2...i_s}}, \quad (k = 1, 2, ..., m)$$

Noether's theorem

$$\frac{\delta}{\delta u^j}\left(XF + FD_i\xi^i\right) = X(\frac{\delta F}{\delta u^j}) + \frac{\delta F}{\delta u^k}\left(\frac{\partial \eta^k}{\partial u^j} - \frac{\partial \xi^i}{\partial u^j}u_i^k + \delta_{kj}D_i\xi^i\right)$$

Noether's theorem

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If

$$XF + FD_i\xi^i = D_iB^i, \quad \frac{\delta F}{\delta u^j} = 0$$

then

$$X\left(\frac{\delta F}{\delta u^j}\right)_{|\frac{\delta F}{\delta u}=0} = 0$$

Variational (divergent) symmetry is a symmetry

Shmyglevskii's approach. Gas dynamics

Terent'ev&Shmyglevskii (1980)

Shmyglevskii's approach. Gas dynamics

Terent'ev&Shmyglevskii (1980)
Bateman (1929), Ito (1955), Shmyglevskii (1980)

$$\mathcal{L} = \rho\left(\frac{u^2}{2} + \dot{\varphi} + S\dot{\mu}\right) - \rho U(\rho, S)$$

where
φ and μ play the roles of Lagrange's multipliers.
S is the entropy ,
$U(\rho, S)$ is the internal energy

Shmyglevskii's approach. Fluids with internal inertia

$$\mathcal{L} = \rho\left(\frac{u^2}{2} + \dot{\varphi} + S\dot{\mu}\right) - W(\rho, \dot{\rho}, S).$$

Euler–Lagrange equations

$$u = -\nabla\varphi - S\nabla\mu + \rho^{-1}W_{\dot{\rho}}\nabla\rho, \quad \dot{\mu} = \rho^{-1}W_S,$$
$$\frac{u^2}{2} + \dot{\varphi} + S\dot{\mu} = W_\rho - \frac{\partial W_{\dot{\rho}}}{\partial t} - div(W_{\dot{\rho}}u),$$
$$\frac{\partial\rho}{\partial t} + div\,(\rho u) = 0, \quad \frac{\partial(\rho S)}{\partial t} + div\,(\rho S u) = 0,$$

Shmyglevskii's approach. Fluids with internal inertia

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Euler–Lagrange equations

$$u = -\nabla\varphi - S\nabla\mu + \rho^{-1}W_{\dot{\rho}}\nabla\rho, \quad \dot{\mu} = \rho^{-1}W_S,$$
$$\frac{u^2}{2} + \dot{\varphi} + S\dot{\mu} = W_\rho - \frac{\partial W_{\dot{\rho}}}{\partial t} - div(W_{\dot{\rho}}u),$$
$$\frac{\partial\rho}{\partial t} + div\,(\rho u) = 0, \quad \frac{\partial(\rho S)}{\partial t} + div\,(\rho S u) = 0,$$

$$\dot{\rho} + \rho u_x = 0, \qquad \rho\dot{u} + p_x = 0, \qquad \dot{S} = 0,$$
$$p = \rho\frac{\delta W}{\delta\rho} - W = \rho\left(W_\rho - (W_{\dot{\rho}} - div\,(uW_{\dot{\rho}})\right) - W,$$

Shmyglevskii's approach. Example of a conservation law

$$W = \rho^{-3}\dot{\rho}^2\eta, \quad X = t\partial_t - u\partial_u$$

$$\frac{\partial}{\partial t}C^1 + \frac{\partial}{\partial x}C^2 = 0,$$

Shmyglevskii's approach. Example of a conservation law

$$W=\rho^{-3}\dot{\rho}^2\eta,\quad X=t\partial_t-u\partial_u$$

$$\frac{\partial}{\partial t}C^1+\frac{\partial}{\partial x}C^2=0,$$

$$C^1=t\rho\frac{u^2}{2}-t\rho^{-3}\eta\dot{\rho}^2+\underline{\rho(\varphi+\eta\mu)},$$

$$C^2=-t\rho\frac{u^3}{2}-tu\eta\rho^{-3}\dot{\rho}^2-4tu^2\eta\rho^{-3}\dot{\rho}\rho_x+4tu^2\eta\rho^{-2}\rho_{tx}-2tu^2\rho^{-2}\dot{\rho}\eta_x$$
$$+2tu^2\rho^{-2}\dot{\rho}\eta\eta_x+2tu\rho^{-2}\dot{\rho}u_x\eta-2tu^2\rho^{-2}u_x\eta\rho_x+2tu\rho^{-2}\eta\rho_{tt}$$
$$+2tu\rho^{-2}\eta u_t\rho_x+2tu^3\rho^{-3}\eta\rho_{xx}+\underline{\rho u(\varphi+\eta\mu)}.$$

Ibragimov's approach

Ibragimov Conservation law's Theorem

Consider a system of m equations

$$F_\alpha(x, u, u_{(1)}, u_2, \dots, u_{(s)}) = 0, \qquad \alpha \quad = 1, \dots, m \tag{1}$$

with n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$. The adjoint system

$$F^*_\alpha(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \dots, u_{(s)}, u_{(s)}) \equiv \frac{\delta \mathcal{L}}{\delta u^\alpha} = 0 \tag{2}$$

inherits the symmetries of the system (1), where

$$\mathcal{L} = v^\beta F_\beta(x, u, u_{(1)}, \dots, u_{(s)})$$

Ibragimov Conservation law's Theorem (continue)

If system (1) admits a point transformation group with a generator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{3}$$

then also the adjoint system (2) admits the operator (3). Then the quantities

$$C^i = v^\beta \left[\xi^i F_\beta + (\eta^\alpha - \xi^j u_j^\alpha)\frac{\partial F_\beta}{\partial u_i^\alpha} \right] \quad , \quad i = 1, \ldots, n \tag{4}$$

furnish a conserved vector $C = (C^1, \ldots, C^n)$ for the system (1).

Ibragimov's approach. Example of a conservation law

The formal Lagrangian

$$\mathcal{L} = (R + \frac{u^2}{2})(\dot{\rho} + \rho u_x) + U(u_t + uu_x + \rho^{-1} p_x) + P\dot{\eta}$$

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Example of multipliers

$$U = \rho u, \quad P = W_\eta - W_{\dot{\rho}\eta}(\dot{\rho} + \rho u_x), \quad R = \frac{\delta W}{\delta \rho} + W_{\dot{\rho}\eta}\dot{\eta}$$

Ibragimov's approach. Example of a conservation law

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$$U = \rho u, \quad P = W_\eta - W_{\dot{\rho}\eta}(\dot{\rho} + \rho u_x), \quad R = \frac{\delta W}{\delta \rho} + W_{\dot{\rho}\eta}\dot{\eta}$$
$$W = \rho^{-3}\dot{\rho}^2\eta, \quad X = t\partial_t - u\partial_u$$

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$$\mathcal{L} = (R + \frac{u^2}{2})(\dot{\rho} + \rho u_x) + U(u_t + uu_x + \rho^{-1} p_x) + P\dot{\eta}$$

Example of multipliers

$$U = \rho u, \quad P = W_\eta - W_{\dot{\rho}\eta}(\dot{\rho} + \rho u_x), \quad R = \frac{\delta W}{\delta \rho} + W_{\dot{\rho}\eta}\dot{\eta}$$

$$W = \rho^{-3}\dot{\rho}^2\eta, \quad X = t\partial_t - u\partial_u$$

$$C^1 = -tu\rho u_t - \rho u^2 - \frac{1}{2}tu^2\dot{\rho} + \frac{1}{2}tu^3\rho_x + 11t\eta\rho^{-4}\dot{\rho}^3 - 11tu\eta\rho^{-4}\dot{\rho}^2\rho_x$$
$$+8tu\eta\rho^{-3}\dot{\rho}\rho_{tx} + tu\eta_x\rho^{-3}\dot{\rho}^2$$

$$C^2 = -\frac{3}{2}tu^2\rho u_t - \frac{3}{2}u^2\rho - \frac{1}{2}tu^3\dot{\rho} + \frac{1}{2}tu^4\rho_x - 5t\eta\rho^{-3}\dot{\rho}^2 u_t + 23tu\eta\rho^{-4}\dot{\rho}^3$$
$$+8tu\eta\rho^{-3}\dot{\rho}u_t\rho_x - 47tu^2\eta\rho^{-4}\dot{\rho}^2\rho_x + 8tu^2\eta\rho^{-3}\rho_x^2 u_t + 11u\eta\rho^{-3}\dot{\rho}^2$$
$$+5tu^2\rho^{-3}\dot{\rho}^2\eta_x - 8tu^3\rho^{-3}\dot{\rho}\rho_x\eta_x + 8tu^2\eta\rho^{-3}\rho_x(\rho_{tt} + u\rho_{tx}) + 24tu^3\eta\rho^{-4}\dot{\rho}\rho_x^2$$

Equations in Lagrangian variables

$$x = \Phi(t, X)$$

Here $x \in \mathcal{D}(t)$ is called a trajectory of the point $X \in \mathcal{D}(t_0)$, the deformation gradient

$$F = \frac{\partial x}{\partial X} = \frac{\partial \Phi(t, X)}{\partial X}$$

The functions $u(t, x)$, $\rho(t, x)$, and $S(t, x)$ in the Eulerian coordinate system are written through the Lagrangian coordinates

$$\rho(t, \Phi(t, X))\Phi_X(t, X) = \rho_0(X), \qquad S(t, \Phi(t, X)) = S_0(X),$$
$$u(t, \Phi(t, X)) = \Phi_t(t, X)$$

Lagrangian

In Eulerian variables

$$\mathcal{L}^E(\rho, u, \dot{\rho}, S) = \rho \frac{u^2}{2} - W(\rho, \dot{\rho}, S)$$

Lagrangian

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In Lagrangian variables

$$\mathcal{L}^L(\rho_0, \Phi_t, \Phi_X, \Phi_{tx}, S_0) = \rho_0\frac{\Phi_t^2}{2} - \Phi_X W\left(\frac{\rho_0}{\Phi_X}, -\frac{\Phi_{tx}}{\Phi_x^2}, S_0\right)$$

Euler-Lagrange equations

The Euler-Lagrange equation

$$\frac{\delta}{\delta\Phi}\mathcal{L}^L = 0$$

$$\frac{\delta}{\delta\Phi} = \frac{\partial}{\partial\Phi} - D_t\frac{\partial}{\partial\Phi_t} - D_X\frac{\partial}{\partial\Phi_X} + D_t^2\frac{\partial}{\partial\Phi_{tt}} + D_tD_X\frac{\partial}{\partial\Phi_{tX}} + D_X^2\frac{\partial}{\partial\Phi_{XX}}$$

reduces to the moment equation in Eulerian coordinates

$$\rho\dot{u} + p_x = 0, \quad p = \rho\frac{\delta W}{\delta\rho} - W,$$

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The constraints

$$\rho_0 = \rho_0(X), \quad S_0 = S_0(X)$$

become the equations

$$\dot{\rho} + \rho u_x = 0, \quad \dot{S} = 0$$

The case $W_{\dot{\rho}} = 0$

$$\Phi_X^3 \Phi_{tt} - W_{\rho\rho}\rho_0 \Phi_{XX} - W_S \frac{\Phi_X^3 S_0'}{\rho_0} + W_{\rho\rho}\rho_0' \Phi_X + W_{\rho S}\rho_0 \Phi_X^2 S_0' = 0,$$

Equivalence transformation

$$\hat{X} = g(X), \ \ \hat{\rho}_0 = \alpha(X)\rho_0(X),$$

where

$$\alpha(X) = h'(g(X)),$$

$X = h(\hat{X})$ is the inverse function of $g(X)$:

$$h(g(X)) = X.$$

Gas dynamics equations in Lagrangian mass coordinate

In particular

$$\rho_0(X)\alpha(X) = 1$$

gas dynamics equations become

$$\Phi_{tt} + p_X = 0,$$

$$p = \rho\frac{\delta W}{\delta\rho} - W$$

Some review of group properties

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Group classification

$$u_{tt} = D_x\varphi, \quad \varphi = \varphi(x, u_x)$$

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- $\varphi = -b(x) u_x^\gamma, \quad \gamma > 1$
  (Polytropic gas: Andreev, Kaptsov, Rodionov, Pukhnachev (1998))
- $\varphi = c(x) u_x, \quad (\gamma = 1)$
  (Grimshaw, Pelinovskii, Pelinovskii (2011))
- $\varphi = \varphi(x, u_x)$
  (C.Kaewmanee, S.V.Meleshko, S.G.Gavrilyuk (2015))

Invariant solutions

Solutions of partial differential equations with two independent variables (t, x)

$$F(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0$$

invariant with respect to 1-dimensional Lie algebra are reduced to a system of ordinary differential equations.

Methods for constructing exact solutions

$$u_{tt} = u_{xx} \Longleftrightarrow \begin{cases} u_t = v_x, \\ v_t = u_x, \end{cases} \qquad \begin{cases} u = f(x-t) + g(x+t), \\ v = -f(x-t) + g(x+t) \end{cases}$$

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Assumptions:

a) Method of generate hodograph $\Phi(u, v) = 0$:

$$v = \phi(u) \quad \begin{cases} v_t = \phi' u_t, \\ v_x = \phi' u_x, \end{cases} \quad \begin{cases} u_t = u_x \phi', \\ u_t \phi' = u_x, \end{cases} \Longrightarrow (\phi')^2 = 1$$

$$\phi' = -1 \Longrightarrow u_t = -u_x, \Longrightarrow u = f(x-t)$$

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b) Method of differential constraints $\Phi(u, v, u_t, u_x, v_t, v_x, x, t) = 0$:

$$u_x + v_x = 0 \Longrightarrow \begin{cases} u = f(x-t), \\ v = -f(x-t). \end{cases}$$

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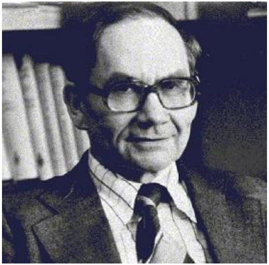
c) Group analysis:

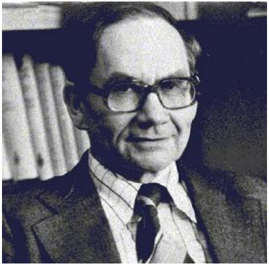
$t\partial_t + x\partial_x$:	$xu_x + tu_t = 0$
$D\partial_x + \partial_t$:	$Du_x + u_t = 0$
∂_x :	$u_x = 0$

Method of differential constraints



N.N. Yanenko (1921-1984)

Method of differential constraints



N.N. Yanenko (1921-1984)

$$S_i(x, u, p) = 0, \quad (i = \overline{1, s}). \tag{5}$$

$$\Phi_k(x, u, p) = 0, \quad (k = \overline{1, q}). \tag{6}$$

Method of differential constraints

A solution of system

$$S_i(x, u, p) = 0, \quad (i = \overline{1, s}),$$

satisfying differential constraints

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Invariant solutions and Lie-Bäcklund symmetries (involutive),
Nonclassical, weak and conditional symmetries

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Invariant solutions and Lie-Bäcklund symmetries (involutive), Nonclassical, weak and conditional symmetries

A.F.Sidorov, V.P.Shapeev and N.N.Yanenko "Methods of differential constraints and its applications in gas dynamics", (Nauka, 2005)

S.V.Meleshko "Methods for constructing exact solutions of PDEs", (Springer, 2005)

One–dimensional gas flows

$$\begin{cases} u_t + uu_x + \rho^{-1} p_x = 0, \\ \rho_t + u\rho_x + \rho u_x = 0, \\ p_t + up_x + A(\rho, p) u_x = 0. \end{cases}$$

$\tau = 1/\rho, \quad c^2 = A\tau, \quad \alpha = \pm c, \quad A = \gamma p, \quad \gamma > 1$

$$S \equiv L\mathbf{u}_t + \Lambda L\mathbf{u}_x = 0,$$

$$L = \begin{pmatrix} 0 & -A & \rho \\ \rho c & 0 & 1 \\ -\rho c & 0 & 1 \end{pmatrix}, \; \Lambda = \begin{pmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{pmatrix}.$$

Riemann (simple) waves: $u = u(\rho), \quad p = p(\rho) \Longrightarrow$ entropy is constant

$$p_x - \rho\alpha^2 \rho_x = 0, \quad \rho u_x + \alpha \rho_x = 0,$$

Solutions with functional arbitrariness

Theorem. The general solution for differential constraints

$$p_x - \alpha^2 \rho_x = \psi, \quad u_x + \rho^{-1} \alpha u_x = \phi$$

1. $\psi = 0$ (isentropic flow)
  1.a Riemann wave $\phi = 0$
  1.b $\gamma = 3$
  1.c $\gamma = 5/3$ (one-atomic gas)
2. $\psi \neq 0$ (nonisentropic flow)

$$\psi = k\rho^{\beta_1} p^{\beta_2}, \quad \phi = -\frac{3\gamma}{(3\gamma - 1)\alpha\rho}\psi$$

$$\beta_1 = 1 - \frac{\gamma}{(3\gamma - 1)}, \quad \beta_2 = 1 + \frac{1}{(3\gamma - 1)}$$

Integration of generalized simple waves

$$\begin{cases} \dfrac{dx}{dt} = u - \alpha \\ \dfrac{d\rho}{dt} = -3\gamma k p^{\beta_1 - 1/2} \rho^{\beta_2 + 1/2}, \end{cases}$$

$$\frac{dp}{d\rho} = \frac{p}{3\rho}, \quad \frac{du}{d\rho} = \frac{\alpha}{3\gamma\rho}$$

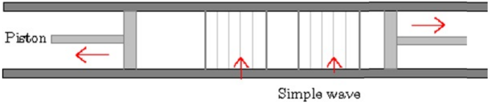
Rarefaction waves

(p, u)**–diagram**:

$$p_\xi - \rho\alpha^2\rho_\xi = 0, \quad \rho u_\xi + \alpha\rho_\xi = 0,$$

This gives **nonisentropic rarefaction** waves.

Interaction of two generalized simple waves



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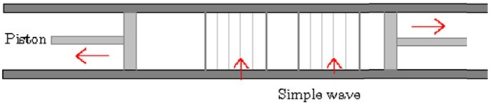

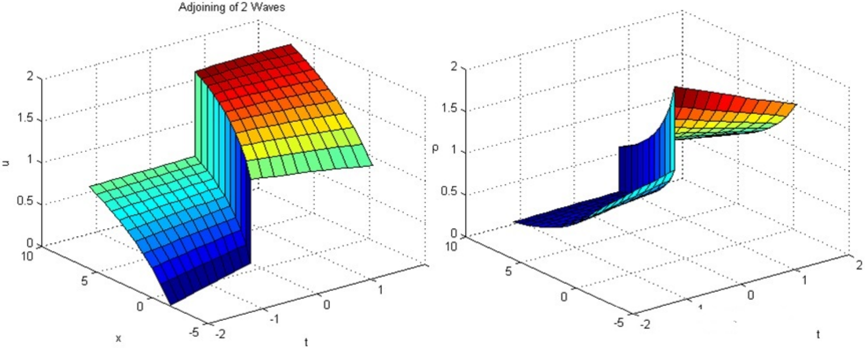

- Integration along a shock wave.
- Integration along characteristics.

The equation $u_{tt} = D_x\varphi, \quad (\varphi = \varphi(x, u_x))$

Potential form

$$u_t = h_x, \;\; h_t = \varphi$$

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$$h_x = g(t, x, u, h, u_x)$$

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Determining equation

$$u_{xx}(\varphi_{u_x} - g_{u_x}^2) = g_t + g_{u_x} g_x + u_x g_u g_{u_x} - \varphi_x + g_h g_{u_x} g + g_h \varphi + g_u g$$

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$$g_{u_x} = G, \ \ g_t = -g_h(\varphi + gG) - g_u(g + Gu_x) - g_x G + \varphi_x$$

Polytropic gas

$$\varphi(x, u_x) = -b(x)u_x^{-\gamma}, \quad G^2 = b\gamma u_x^{-(\gamma+1)}$$

$$g_{u_x} = G, \quad g_u = 0, \quad g_t = 0, \quad g_x = g_h(u_x^{-\gamma} bG^{-1} - g) - u_x^{-\gamma} b_x G^{-1}$$

$$(3\gamma - 1)bb_{xx} - 3\gamma b_x^2 = 0$$

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Polytropic gas

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- Solve the system of ordinary differential equations ($v = u_x$)

$$\frac{dx}{dt} = -G, \quad \frac{dv}{dt} = -3\frac{b'G}{b(3\gamma - 1)}v,$$

$$\frac{du}{dt} = -\frac{b'}{b(3\gamma - 1)}h + g_1 - Gv, \quad \frac{dh}{dt} = G(\frac{b'}{b(3\gamma - 1)}h - g_1) - bv^{-\gamma}$$

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$$G = \alpha\,(b\gamma)^{1/2}\,v^{-\frac{(\gamma+1)}{2}}, \quad \alpha = \pm 1, \quad g_1 = kb^{\frac{1}{3\gamma-1}} - 2\alpha\frac{(\gamma b)^{1/2}}{(\gamma - 1)}v^{(1-\gamma)/2}$$

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- The first step is group classification of these equations. It is completed for $W_{\dot{\rho}} = 0$
- Attract attention to the method of differential constraints

Acknowledgements

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Thank you very much for your attention!